Lectures notes on affine transformations (with exercises)
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Corrections and comments are welcome!

1. Affine transformations

Recall that the 2×2 matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

rotates the plane $\mathbb{R}^2$ counter-clockwise by the angle θ around the origin, while the matrix

$$M_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{2}$$

reflects the plane across the line passing through the origin at an angle θ to the x-axis.

How can we compute rotations around other points in the plane besides the origin? Or reflections across lines that do not pass through the origin?

First consider rotations: To rotate around a point $\mathbf{b} \in \mathbb{R}^2$ by an angle θ, we would first translate $\mathbf{b}$ to the origin (that is, translate by $-\mathbf{b}$, then rotate as necessary around the origin (using the matrix (1)), then translate the origin back to $\mathbf{b}$.

More specifically, let $R_{\theta,\mathbf{b}}$ denote the counter-clockwise rotation of the plane by angle θ around the point $\mathbf{b}$. For all $X \in \mathbb{R}$, we have

$$R_{\theta,\mathbf{b}}(X) = R_\theta(X - \mathbf{b}) + \mathbf{b} = R_\theta X - R_\theta \mathbf{b} + \mathbf{b} = R_\theta X + (I - R_\theta)\mathbf{b}, \tag{3}$$

where I is the 2×2 identity matrix.

Similarly, to reflect across a line through the point $\mathbf{b}$ that makes an angle θ with the horizontal, we use the transformation

$$M_{\theta,\mathbf{b}}(X) = M_\theta X + (I - M_\theta)\mathbf{b}. \tag{4}$$

Note that these transformations are *not* linear when $\mathbf{b} \neq 0$. They belong to a larger class of functions on vector spaces called *affine transformations.* Let V and W denote vector spaces. An affine transformation is function $f : V \longrightarrow W$ of the form

$$f(X) = T(X) + \mathbf{b},$$

where $T : V \longrightarrow W$ is *linear* and $\mathbf{b} \in W$ is a constant vector in W. In other words, an affine transformation is a combination of a linear transformation and a *translation* by some constant vector $\mathbf{b}$.

Just as linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are represented by an $m \times n$ matrix, an affine transformation f from $\mathbb{R}^n$ to $\mathbb{R}^m$ will have the form

$$f(X) = AX + \mathbf{b},$$

for some $m \times n$ matrix A and a constant $m \times 1$ column vector $\mathbf{b} \in \mathbb{R}^m$.

2. Homogeneous coordinates

In this section we will focus on $\mathbb{R}^2$, although entirely analogous methods can be used for $\mathbb{R}^n$ as well.

In order to represent affine transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ by matrices, we switch from Cartesian coordinates (used until now) to *homogeneous coordinates.* We do this by representing the point (x, y) in $\mathbb{R}^2$ using the 3-dimensional vector $(x, y, 1)$. In other words, we identify the plane $\mathbb{R}^2$ with the plane $z = 1$ in $\mathbb{R}^3$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

One advantage to using homogeneous coordinates is that translations can be computed using matrix multiplication. To see this, note that

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + b_1 \\ y + b_2 \\ 1 \end{bmatrix}$$

So the translation $X \to X + \mathbf{b}$ is represented in homogeneous coordinates by multiplication by the matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\mathbf{b} = (b_1, b_2)$.

Meanwhile, linear transformations are also easily represented by matrices in homogeneous coordinates. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then the linear transformation $X \to AX$ on $\mathbb{R}^2$ is represented in homogeneous coordinates by the transformation:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Putting these two observations together, we find that the affine transformation

$$f(X) = AX + \mathbf{b}$$

on $\mathbb{R}^2$ is represented in homogeneous coordinates as left matrix multiplication:

$$f(X) = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{5}$$

3. Examples of rotations and reflections.

Example 1. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote the rotation of the plane by the counter-clockwise angle $\frac{\pi}{3}$ around the point $(2, -3)$.

(a) Find the matrix that represents this rotation in homogeneous coordinates.

(b) Compute $F(0, 4)$.

Solution: To solve part **(a)**, we first translate by $-\mathbf{b} = (-2, 3)$ to move the point $(2, -3)$ to the origin. Then we rotate around the origin by angle $\pi/3$. Then we translate by $\mathbf{b} = (2, -3)$ to move the origin back to $(2, -3)$. This has the net effect of rotating around the point $(2, -3)$. The composition of these three operations will be represented by the matrix product:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} & \frac{2-3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{-3-2\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

∎

Note well the **order** of the matrices in this product! The first operation is on the right, and subsequent operations correspond to multiplying by subsequent matrices on the left side.

This completes part **(a)**.

To solve part **(b)**, use the matrix from part **(a)** to compute:

$$\begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} & \frac{2-3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{-3-2\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2-7\sqrt{3}}{2} \\ \frac{1-2\sqrt{3}}{2} \\ 1 \end{bmatrix},$$

so that $F(0, 4) = (\frac{2-7\sqrt{3}}{2}, \frac{1-2\sqrt{3}}{2})$.

Alternative Solution to Example 1(a): We can use the formula (3) instead. Recall from (3) that

$$R_{\pi/3,(2,-3)}(X) = R_{\pi/3}X + (I - R_{\pi/3})\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} X + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} X + \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} X + \begin{bmatrix} \frac{2-3\sqrt{3}}{2} \\ \frac{-2\sqrt{3}-3}{2} \end{bmatrix}$$

Applying the formula (5) we conclude that

$$f(X) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{2-3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{-2\sqrt{3}-3}{2} \\ 0 & 0 & 1 \end{bmatrix} X$$

once again. ∎

Example 2. Find the matrix in homogeneous coordinates that reflects the plane across the line $x = 2$.

Solution: First, translate the vertical line $x = 2$ by the vector $-\mathbf{b} = (-2, 0)$ to the y-axis. Then reflect acorss the y-axis. Then translate back! To do this, compute the matrix product:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note once again the **order** of the matrix multiplication. The matrix in the middle of the triple product negates the x-coordinate, providing the reflection of $\mathbb{R}^2$ across the y-axis (in homogeneous coordinates). ∎

Alternative Solution: Using the equations (2) and (4) we obtain

$$f(X) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X + \left(I - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{b} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Applying the formula (5) we conclude that

$$f(X) = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X$$

once again. ∎

4. General affine transformations

Recall that if $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a *linear* transformation, then T is determined by its action on any *basis* of $\mathbb{R}^2$. That is, if you know $T(X_1)$ and $T(X_2)$ for some basis $\{X_1, X_2\}$ of $\mathbb{R}^2$, then you can find the matrix of T.

An affine transformation on $\mathbb{R}^2$ is similarly determined by its action on any *three* non-collinear points, that is, any three points that form a proper triangle. [1].

Specifically, let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an affine transformation. Suppose you are given three non-collinear points X_1, X_2, X_3 in $\mathbb{R}^2$ and suppose you also know the values $Y_1 = f(X_1)$, $Y_2 = f(X_2)$, and $Y_3 = f(X_3)$.

Since f is affine, it has the form $f(X) = AX + \mathbf{b}$ for some 2×2 matrix A and some vector $\mathbf{b} \in \mathbb{R}^2$. In homogeneous coordinates this means that

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{b} \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} X_i \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} Y_i \\ \hline 1 \end{array}\right]$$

for $i = 1, 2, 3$. In other words,

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{b} \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} X_1 & X_2 & X_3 \\ \hline 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc} Y_1 & Y_2 & Y_3 \\ \hline 1 & 1 & 1 \end{array}\right]$$

so that

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{b} \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} Y_1 & Y_2 & Y_3 \\ \hline 1 & 1 & 1 \end{array}\right] \left[\begin{array}{ccc} X_1 & X_2 & X_3 \\ \hline 1 & 1 & 1 \end{array}\right]^{-1}$$

It will be possible to take necessary matrix inverse in the equation above, because we made sure the vectors X_i were not collinear. (It's a good exercise to verify this assertion.)

Example: Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an affine transformation such that

$$\begin{aligned} f(1,1) &= (3,-4) \\ f(0,2) &= (-1,-1) \\ f(-1,1) &= (1,0) \end{aligned}$$

(a) Find the matrix that represents f in homogeneous coordinates.

(b) Compute $f(6,-8)$.

[1] More generally, an affine transformation on $\mathbb{R}^n$ is uniquely determined by its action on any $n+1$ non-coplanar points.

Solution: To solve part **(a)** we compute

$$\begin{bmatrix} 3 & -1 & 1 \\ -4 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -4 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

For part **(b)**, use the matrix above to compute:

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -8 \\ 1 \end{bmatrix} = \begin{bmatrix} 35 \\ -23 \\ 1 \end{bmatrix},$$

so that $f(6, -8) = (35, -23)$.

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Exercises:

1. Compute the matrix that rotates the plane $\mathbb{R}^2$ counter-clockwise by the angle $\frac{\pi}{4}$ around the point $(3, 2)$ using homogeneous coordinates.

2. Compute the matrix that rotates the plane $\mathbb{R}^2$ *clockwise* by the angle $\frac{\pi}{4}$ around the point $(3, 2)$ using homogeneous coordinates.

3. Compute the matrix that reflects the plane $\mathbb{R}^2$ across the line $y = -4$ using homogeneous coordinates.

4. Compute the matrix that dilates the plane $\mathbb{R}^2$ by 3 from the point (7, 2) using homogeneous coordinates.

5. Find the image of the point $(20, 9)$ under the counter-clockwise rotation by $\pi/3$ around the point (2, -3).
(**Hint:** See the example presented earlier in these notes.)

6. Find the image of the point $(-4, 1)$ under the rotation of Exercise **1.** above.

7. Find the image of the point $(-4, 1)$ under the rotation of Exercise **3.** above.

8. An affine transformation $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ maps the three points

$$(1, 2) \qquad (2, 1) \qquad (3, 3)$$

to the points

$$(0, -2) \qquad (1, -1) \qquad (4, 0)$$

respectively. Find the matrix that represents the transformation F in homogeneous coordinates.

9. Compute $F(6, -3)$, where F is the affine transformation of Exercise **8.** above.

10. In analogy to the 2-dimensional case, homogeneous coordinates for $\mathbb{R}^3$ use 4-dimensional vectors with final coordinate 1 and suitably arranged 4×4 matrices.

Problem: Compute the 4×4 matrix that reflects 3-dimensional space $\mathbb{R}^3$ across the plane $z = 6$ using homogeneous coordinates.

Answers to Selected Exercises:

1. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 3 - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 - \frac{5}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 - \frac{5}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 - \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}$

(Note: This corresponds to a "counter-clockwise" rotation by $-\frac{\pi}{4}$ in the formula (1).

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 & 0 & -14 \\ 0 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ 1 \end{bmatrix}$

8. The matrix A of this affine transformation satisfies:

$$A \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

so that

$$A = \begin{bmatrix} 0 & 1 & 4 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 4 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & -3 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

10. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$