Continuous and profinite combinatorics

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1. Geometric probability

Gian-Carlo Rota believed that mathematics is a unity, in the deep sense that the same themes recur – as analogies – in its many branches. Thus, it comes as no surprise that Rota perceived combinatorial themes in continuous mathematics (as well as the other way around). In the preface to his book *Introduction to Geometric Probability* [10], written with the first author, Rota suggested only partly in jest that the field of geometric probability be renamed "continuous combinatorics". Two of the papers reprinted in this chapter paper belong to the research program described in [10].

Perhaps the central idea behind Rota's "continuous combinatorics" is the analogy between counting and measure, especially measures that are invariant with respect to some symmetry or group action. Here, the word "measure" is used in the broadest sense to include finitely additive measures which may admit no countably additive extension. These finitely additive measures, also called *valuations*, provide the intermediate hues in a spectrum of set functionals that extends from the purely discrete (such as lattice point enumerators and the Euler characteristic) to the analytic measures of Lebesgue theory.

Although far more attention has been given in the last century to the two extreme cases (combinatorics and real analysis), constructions in convex and integral geometry going back as far as Minkowski offer a panoply of invariant valuations that are neither wholly analytic nor combinatorial in nature. Functionals on polytopes and convex sets such as the mean width, projection functions onto flats, and more general families of intrinsic volumes [10, 12], mixed volumes [18], and dual mixed volumes [11], provide examples whose fundamental properties are still poorly understood as compared to Lebesgue measure and simple counting. It was Rota's contention that the best way to develop a comprehensive theory of these intermediate functionals is to determine how they connect analogous structures observed in combinatorics and real analysis, structures that are most evident in the contexts of combinatorial and analytic convex geometry.

The analogies between the intrinsic volumes or *Quermassintegrals* (characterized by Hadwiger [6] as the fundamental valuations invariant under rigid motions), the Ehrhart coefficients of lattice polytopes (which are affine unimodular invariants later characterized by Betke and Kneser [1]), and fundamental families of enumerative functionals on simplicial complexes and finite vector spaces (for example, face and subspace enumerators [5, 8, 9, 10]), provide further evidence that a comprehensive theory of invariant set functionals is waiting in the wings.

The motivation for the paper "Totally invariant set functions of polynomial type", written with Beifang Chen, can be found in Problem Five in [14]. This problem, "Set functions on convex bodies", is to prove the "correct" statement of the conjecture:

> *Subject to technical assumptions,* every invariant set function defined on convex bodies is associated with a symmetric function.

This conjecture is motivated by the observation that the intrinsic volumes of a rectangular parallelotope are elementary symmetric functions of the lengths of its sides (see [10] and the expository papers [15, 16]). In the reprinted paper, Chen and Rota proved a version of this conjecture, but, as Rota himself put it in [14], the technical assumptions are "preposterous." Rota's fifth problem awaits the development of a truly combinatorial approach to defining intrinsic volumes and other set functions on convex bodies invariant under Euclidean motions.

Sperner's theorem, that the maximum size of an antichain in the Boolean algebra of subsets of an n-element set equals the maximum binomial coefficient $\binom{n}{k}$, where k is $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$, began extremal set theory. An analogue holds for the lattice of subspaces of vector space over a finite field. In the paper "A continuous analogue of Sperner's theorem", Rota and Klain proved an analogue for the lattice of subspaces of a finite-dimensional real vector space. The tools for doing this are a continuous analogue of the LYM-inequality and careful choices of normalizations for measures on Grassmannians, inspired in turn by normalizations that transform the Quermassintegrals into the intrinsic volumes on convex bodies. Variations on this theme can be found in [10].

2. Profinite limits

Perhaps the most striking example of a profinite limit is due to von Neumann [13] (see also [2], p. 237). Von Neumann observed that there is a natural embedding of projective geometries over a finite field $\mathrm{GF}(q)$ of order q which doubles the rank. Thinking of $\mathrm{PG}(n, q)$ as the lattice of subspaces of an n-dimensional subspace over $\mathrm{GF}(q)$, this embedding

$$\mathrm{PG}(2^m, q) \to \mathrm{PG}(2^m, q) \times \mathrm{PG}(2^m, q) \to \mathrm{PG}(2^{m+1}, q)$$

is given by

$$a \mapsto (a, a) \mapsto (a, 0) \vee (0, a).$$

Taking the completion of the profinite limit of the directed system

$$\mathrm{PG}(1, q) \to \mathrm{PG}(2, q) \to \mathrm{PG}(4, q) \to \ldots \to \mathrm{PG}(2^m, q) \to \mathrm{PG}(2^{m+1}, q) \to \ldots,$$

von Neumann obtains the "pointless" *continuous geometry* $\mathrm{CG}(q)$ *over the finite field of order* q in which there are subspaces of "dimension" r for every real number r in the unit interval $[0, 1]$.

This example must have fascinated Rota, and in his paper [5] with Jay Goldman, he proposed using it to study the theory of infinite q-series. As Rota has shown (see [5] or [14]), one can prove finite q-identities by finding bijections between objects (such as subspaces, bases, and linear transformations) defined on finite vector or projective spaces. However, identities between infinite q-series (such as the Rogers-Ramanujan identities) cannot be directly proved in this way. Rota suggested using the continuous geometries $\mathrm{CG}(q)$ for this purpose. He also suggested that there is a q-analog of the Poisson process on $\mathrm{CG}(q)$. Another natural problem is to find an analogue of Sperner's theorem for $\mathrm{CG}(q)$.

Rota suggested that von Neumann's construction can also be done with lattices of partition of finite sets. This is done in [3]. See also [7]. However, the next step ([17], Section 4), that of using such continuous partition lattice to study entropy, has yet to be taken.

Von Neumann's construction is one example of a general method for constructing an infinite object from a directed system of finite objects. Rota was fascinated by such "profinite" objects; they seem to offer another bridge between the finite and the infinite. Rota wrote only one paper devoted specifically to his ideas on profinite combinatorics. This is the paper "A stochastic interpretation of the Riemann zeta function", written with Kenneth Alexander and Kenneth Baclawski. In this paper, a profinite limit of cyclic groups is used to give an interpretation of zeta functions defined by critical problems. Rota was optimistic – perhaps too optimistic – about this probabilistic approach. His ultimate aim is to obtain a bijective proof of the functional equation. However, among a host of technical problems, this would require a compatible stochastic interpretation of the gamma function. The paper "An example of profinite combinatorics" is extracted from his notes for a lecture to a general audience. It gives an elementary account of the probabilistic interpretation of the Riemann zeta function.

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