Dehn-Sommerville relations for triangulated manifolds

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Abstract Euler's formula is used to derive relations of Dehn-Sommerville type for a triangulated manifold with boundary.

In the present note we derive a family of Dehn-Sommerville type relations for triangulated manifolds with boundary. These relations generalize the classical Dehn-Sommerville equations for spherical simplicial complexes. As a consequence of these relations we obtain formulas for the enumeration of vertices (0-simplices) lying in the relative interior of triangulated manifold with boundary. Analogous enumerative formulas are obtained for higher dimensional faces.

Let M be a compact triangulated manifold with boundary ∂M. Denote by $\mathrm{int}(M)$ the (topological) relative interior of M; that is $\mathrm{int}(M) = M - \partial M$. Denote by $K(M)$ the simplicial complex whose geometric realization is the manifold M. For $k = 0, 1, \ldots, m$ and any subset $L \subseteq K(M)$, denote $f_k(L)$ the number of k-dimensional simplices contained in L. Recall that the *Euler characteristic* $\chi(L)$ of a subcomplex (or any subset) $L \subseteq K(M)$ is defined by

$$\chi(L) = \sum_{i=0}^{m} (-1)^i f_i(L).$$

Our main result is the following.

Theorem 1.1 *Let M be an m-dimensional triangulated manifold with boundary. For $k = 0, \ldots, m$*

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M). \tag{1}$$

Theorem 1.1 can be viewed as an Euler relation for the *valuations* f_k on the lattice of subcomplexes of $K(M)$. For a discussion of valuations, see [5, 7]. This point of view is exploited more extensively for polytopes in [4], and also (in a different way) in [6].

For $\sigma \in K(M)$ define the *star* of σ by

$$\mathrm{St}\ \sigma = \{\tau \in K(M) \mid \sigma \subseteq \tau\}.$$

For $\sigma \in K(M)$, let $\overline{\sigma}$ denote the set of all faces of σ, and define the *closed star* of σ

$$\overline{\mathrm{St}}\,\sigma = \bigcup_{\tau \in \mathrm{St}\ \sigma} \overline{\tau}.$$

Let K^σ denote the subcomplex of $K(M)$ given by

$$K^\sigma = \overline{\mathrm{St}}\,\sigma - \mathrm{St}\ \sigma$$

The following lemma will be used in the proof of Theorem 1.1.

Lemma 1.2 *If $\sigma \in K(M)$ then*

$$\chi(K^\sigma) = \begin{cases} 1 + (-1)^{m+1} & \text{if } \sigma \not\subseteq \partial M \\ 1 & \text{if } \sigma \subseteq \partial M \end{cases} \tag{2}$$

Proof: This is an immediate consequence of the homological properties of a manifold with boundary, since the geometric realization of $K^\sigma = \overline{\mathrm{St}}\,\sigma - \mathrm{St}\,\sigma$ is homotopy equivalent a closed m-dimensional ball with a point removed. (See, for example, [8].) ■

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1: Since $\overline{\mathrm{St}}\,\sigma$ is star-shaped, we have $\chi(\overline{\mathrm{St}}\,\sigma) = 1$. Since $\overline{\mathrm{St}}\,\sigma = \mathrm{St}\ \sigma \cup K^\sigma$, a *disjoint* union, it follows from Lemma 1.2 that

$$(-1)^m(\chi(\mathrm{St}\ \sigma)) = \begin{cases} 1 & \textit{if } \sigma \not\subseteq \partial M \\ 0 & \textit{if } \sigma \subseteq \partial M \end{cases}$$

for all $\sigma \in K(M)$. It follows that

$$\begin{aligned} f_k(M) - f_k(\partial M) &= \textstyle\sum_{\{\sigma \in K(M)\,:\,\dim(\sigma)=k\}} (-1)^m(\chi(\mathrm{St}\ \sigma)) \\ &= \textstyle\sum_{\{\sigma \in K(M)\,:\,\dim(\sigma)=k\}} \sum_{i=0}^m (-1)^{i+m} f_i(\mathrm{St}\ \sigma) \end{aligned} \tag{3}$$

In the sum (3) a cell $\tau \in K(M)$ of dimension i is counted once for each $\sigma \in K(M)$ such that $\tau \in \mathrm{St}\ \sigma$; that is, once for each k-cell $\sigma \subseteq \tau$. Since each $\tau \in K(M)$ of dimension i contains $\binom{i+1}{k+1}$ faces of dimension k, each face τ of dimension i in $K(M)$ is counted $\binom{i+1}{k+1}$ times in the sum (3). It follows that

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

■

Note also that if $k = 0$ then

$$f_0(M) - f_0(\partial M) = f_0(\mathrm{int}(M)) = (-1)^m (f_0(M) - 2f_1(M) + 3f_2(M) - \cdots), \quad (4)$$

the number of vertices of the triangulation $K(M)$ that lie in the relative interior of the manifold M.

Corollary 1.3 (Classical Dehn-Sommerville Equations) *Suppose that M is an m-dimensional triangulated manifold (without boundary). Then*

$$f_k(M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M),$$

for $k = 0, 1, \ldots, m$.

Note that setting $k = -1$ in Corollary 1.3 yields the Euler formula for M (up to a change of sign). For this reason the Euler characteristic $\chi(M)$ is also sometimes denoted $f_{-1}(M)$. The family of formulas (1.3) applied to a triangulated sphere give the classical Dehn-Sommerville equations for the spherical simplicial complex (see, for example, [1, 2, 9]).

We conclude this section with a relation dual (and equivalent) to that of Theorem 1.1.

Corollary 1.4 *Let M be an m-dimensional triangulated manifold with boundary ∂M. Then*

$$f_k(M) = \sum_{i=k}^{m} (-1)^{i+m-1} \binom{i+1}{k+1} (f_i(M) - f_i(\partial M)),$$

for $k = 0, 1, \ldots, m$.

Proof: Since $\partial(\partial M) = \emptyset$, we can apply Theorem 1.1 (or Corollary 1.3) to the subcomplex that triangulates ∂M (a manifold of dimension $m - 1$) to obtain

$$f_k(\partial M) = \sum_{i=k}^{m-1} (-1)^{i+m-1} \binom{i+1}{k+1} f_i(\partial M). \quad (5)$$

On summing (5) and (1) we obtain

$$f_k(M) = \sum_{i=k}^{m} (-1)^{i+m-1} \binom{i+1}{k+1} (f_i(M) - f_i(\partial M)).$$

■

See any of [1, 2, 3, 9] for a more detailed treatment of the Dehn-Sommerville equations and their application to combinatorial geometry.

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