Name: ______________________ **Math 2220** **Test #2** **Fall 2019**

Read the problems **carefully**. Please **show all work.**
One sheet of notes (maximum size 8.5×11 inches, double-sided) is permitted.
Calculators and other electronic devices are not permitted on this exam.

1. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for $\mathbb{R}^3$, and let

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

Suppose that $||\mathbf{x}|| = 10$, that $\mathbf{x} \cdot \mathbf{u}_1 = 8$, and that $\mathbf{x} \perp \mathbf{u_2}$.
What are the possible values of c_1, c_2, c_3?

Solution: Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set, we have

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever } i \neq j,$$

and

$$\mathbf{u}_i \cdot \mathbf{u}_i = 1 \quad \text{for each } i.$$

It follows that

$$\mathbf{x} \cdot \mathbf{u}_i = c_i \quad \text{for each } i.$$

In particular,

$$c_1 \ = \ \mathbf{x} \cdot \mathbf{u}_1 = 8,$$

and

$$c_1 \ = \ \mathbf{x} \cdot \mathbf{u}_2 = 0, \qquad \text{since } \mathbf{x} \perp \mathbf{u_2}.$$

Since $||\mathbf{x}|| = 10$ and the $\mathbf{u_i}$ are mutually orthogonal, the Pythagorean Theorem applies, and

$$100 \ = \ ||\mathbf{x}||^2 \ = \ \mathbf{x} \cdot \mathbf{x} \ = \ c_1^2 + c_2^2 + c_3^2,$$

whence

$$c_3^2 \ = \ 100 - c_1^2 - c_2^2 \ = \ 100 - 8^2 - 0^2 \ = \ 36,$$

and $c_3 = \pm 6$.

Conclusion:

$$\boxed{c_1 = 8, \ \ c_2 = 0, \ \ c_3 = \pm 6}$$

2. Suppose that a, b, c are constants, and that the matrix

$$A = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$$

is an orthogonal matrix.

What are the possibilities for the matrix A?

List ***all*** possible examples, with the *actual numerical entries* of A in each case, and explain briefly ***why*** your answer is complete and correct.

Solution: If A is an orthogonal matrix, then the first column must be a unit vector. This means that

$$0^2 + b^2 \ = \ 1,$$

so that $b = \pm 1$.

Moreover, the columns of A must be perpendicular, and have dot product zero. This means that

$$0a + bc \ = \ 0,$$

so that $0 = bc = \pm c$, and so $c = 0$.

Finally, the second column must be a unit vector. This means that

$$1 \ = \ a^2 + c^2 \ = \ a^2 + 0^2 \ = \ a^2,$$

so that $a = \pm 1$.

Conclusion:

$$a = \pm 1, \ b = \pm 1, \ c = 0,$$

so that

$$A \in \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

3. Let L be the line passing through the origin and the vector $(1, -1, 1, -1)$ in $\mathbb{R}^4$.

(a) Find the projection matrix for the subspace (line) L.

Solution:

$$\text{Let} \quad A = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{so that} \quad A^T A = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 4,$$

and

$$(A^T A)^{-1} = \frac{1}{4}.$$

The projection matrix Q is now given by

$$Q = A(A^T A)^{-1} A^T = \frac{1}{4} A A^T = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

(b) Find the reflection matrix for that same line L.

Solution: If H is the reflection matrix, then

$$H = 2Q - I,$$

where Q is the projection matrix from part (a) and I is the 4×4 identity matrix. Therefore,

$$H = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}$$

4. Let $\mathbf{v} = (1, 1, 3, 3)$ and let $\mathbf{w} = (2, 1, -2, 0)$. Let $\mathbf{u}$ be the unit vector pointing in the direction of $\mathbf{w}$. Compute the following:

(a) $\mathbf{v} \cdot \mathbf{w} = \boxed{-3}$

(b) $||\mathbf{u}|| = \boxed{1}$

(c) $\mathbf{u} \cdot \mathbf{w} = \boxed{3}$

(d) $\text{Proj}_{\mathbf{u}}(\mathbf{v}) = \boxed{\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, 0\right)}$

(e) If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ then $\cos\theta = -\frac{1}{\sqrt{20}} = -\frac{1}{2\sqrt{5}} = \boxed{-\frac{\sqrt{5}}{10}}$