Integrating functions of irreducible quadratics

Suppose we are given a quadratic that we cannot easily factor, such as

\[ 2x^2 + 12x + 7. \]

We would like to express this function as a sum or difference of squares. The main tool we will use is the fact that

\[(x + a)^2 = x^2 + 2ax + a^2.\]

Returning to the original problem, first factor out the leading coefficient:

\[ 2x^2 + 12x + 7 = 2(x^2 + 6x + \frac{7}{2}). \]

The \(x\)-coefficient 6 now plays the role of \(2a\). This means \(a = 3\) and \(a^2 = 9\), so we add and subtract 9 to obtain

\[ 2x^2 + 12x + 7 = 2(x^2 + 6x + 9 + \frac{7}{2} - 9). \]

It follows that

\[ 2x^2 + 12x + 7 = 2((x + 3)^2 - \frac{11}{2}). \]

This technique allows us to use trig substitutions for integrals where arbitrary quadratics appear. For example, let us compute

\[ \int \frac{1}{2x^2 + 12x + 7} \, dx. \]

By the previous argument, we have

\[ \int \frac{1}{2x^2 + 12x + 7} \, dx = \int \frac{1}{2((x + 3)^2 - \frac{11}{2})} \, dx \]
\[ = \frac{1}{2} \int \frac{1}{(x + 3)^2 - \frac{11}{2}} \, dx. \]

We can now use the trig substitution

\[ x + 3 = \sqrt{\frac{11}{2}} \sec \theta \]
\[ dx = \sqrt{\frac{11}{2}} \sec \theta \tan \theta \, d\theta \]
We now have
\[
\int \frac{1}{2x^2 + 12x + 7} \, dx = \frac{1}{2} \int \frac{\sqrt{\frac{11}{2}} \sec \theta \tan \theta}{\sec^2 \theta - \frac{11}{2}} \, d\theta
\]
\[
= \frac{1}{2} \sqrt{\frac{11}{2}} \int \frac{\sec \theta \tan \theta}{\sec^2 \theta - 1} \, d\theta
\]
\[
= \frac{1}{\sqrt{22}} \int \frac{\sec \theta \tan \theta}{\tan^2 \theta} \, d\theta
\]
\[
= \frac{1}{\sqrt{22}} \int \frac{\sec \theta \tan \theta}{\tan \theta} \, d\theta
\]
\[
= \frac{1}{\sqrt{22}} \int \frac{1}{\cos \theta \sin \theta} \, d\theta
\]
\[
= \frac{1}{\sqrt{22}} \int \csc \theta \, d\theta
\]
\[
= -\frac{1}{\sqrt{22}} \ln |\csc \theta + \cot \theta| + C
\]

Since
\[
x + 3 = \sqrt{\frac{11}{2}} \sec \theta
\]
we have
\[
\sec \theta = \frac{x + 3}{\sqrt{\frac{11}{2}}} = \frac{HYP}{ADJ}.
\]
So \(HYP = x + 3\) and \(ADJ = \sqrt{\frac{11}{2}}\) and
\[
OPP^2 = HYP^2 - ADJ^2 = (x + 3)^2 - \frac{11}{2} = x^2 + 6x + \frac{7}{2}.
\]
Hence,
\[
csc \theta = \frac{HYP}{OPP} = \frac{x + 3}{\sqrt{x^2 + 6x + \frac{7}{2}}}
\]
and
\[
cot \theta = \frac{ADJ}{OPP} = \frac{\sqrt{\frac{11}{2}}}{\sqrt{x^2 + 6x + \frac{7}{2}}}.\]
We conclude that
\[
\int \frac{1}{2x^2 + 12x + 7} \, dx = -\frac{1}{\sqrt{22}} \ln |\csc \theta + \cot \theta| + C
\]
\[
= -\frac{1}{\sqrt{22}} \ln \left| \frac{x+3}{\sqrt{x^2+6x+\frac{7}{2}}} + \frac{\sqrt{11}}{\sqrt{x^2+6x+\frac{7}{2}}} \right| + C
\]

Here is another example:
Let’s compute
\[
\int \frac{1}{x^2 + 8x + 25} \, dx.
\]
To begin, we complete the square:
\[
x^2 + 8x + 25 = (x^2 + 8x + 16) + (25 - 16) = (x + 4)^2 + 9 = (x + 4)^2 + 3^2.
\]
Notice that the adjustment of ‘16’ is obtained by taking the square of half of the \(x\)-coefficient \(\left(\frac{8}{2} = 4\right)\).
We now have
\[
\int \frac{1}{x^2 + 8x + 25} \, dx = \int \frac{1}{(x + 4)^2 + 3^2} \, dx.
\]
Setting
\[
x + 4 = 3 \tan \theta
\]
\[
dx = 3 \sec^2 \theta \, d\theta
\]
it follows that
\[
\int \frac{1}{x^2 + 8x + 25} \, dx = \int \frac{3 \sec^2 \theta}{3^2 \tan^2 \theta + 3^2} \, d\theta = \frac{1}{3} \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} \, d\theta
\]
\[
= \frac{1}{3} \int \frac{\sec^2 \theta}{\sec^2 \theta} \, d\theta = \frac{1}{3} \int \, d\theta = \frac{1}{3} \theta + C
\]
\[
= \frac{1}{3} \arctan \left(\frac{x + 4}{3}\right) + C
\]
Consider the following variation:

\[ \int \frac{3x + 12}{x^2 + 6x + 3} \, dx. \]

Notice that if \( u = x^2 + 6x + 3 \) then \( du = 2x + 6 \). With this in mind, write

\[ \int \frac{3x + 12}{x^2 + 6x + 3} \, dx = 3 \int \frac{x + 4}{x^2 + 6x + 3} \, dx = \frac{3}{2} \int \frac{2x + 8}{x^2 + 6x + 3} \, dx. \]

This still isn't what we want, but \( 2x + 8 = (2x + 6) + 2 \), so

\[ \int \frac{3x + 12}{x^2 + 6x + 3} \, dx = \frac{3}{2} \int \frac{2x + 6}{x^2 + 6x + 3} \, dx + \frac{3}{2} \int \frac{2}{x^2 + 6x + 3} \, dx \]

\[ = \frac{3}{2} \int \frac{2x + 6}{x^2 + 6x + 3} \, dx + \frac{3}{2} \int \frac{2}{x^2 + 6x + 3} \, dx \]

Applying the \( u \)-substitution \( u = x^2 + 6x + 3 \) and \( du = 2x + 6 \) to the first integral implies that

\[ \frac{3}{2} \int \frac{2x + 6}{x^2 + 6x + 3} \, dx = \frac{3}{2} \int \frac{1}{u} \, du = \frac{3}{2} \ln |u| = \frac{3}{2} \ln |x^2 + 6x + 3|, \]

up to a constant term. Our original problem now becomes

\[ \int \frac{3x + 12}{x^2 + 6x + 3} \, dx = \frac{3}{2} \ln |x^2 + 6x + 3| + 3 \int \frac{1}{x^2 + 6x + 3} \, dx \]

For the second integral, note that

\[ x^2 + 6x + 3 = x^2 + 6x + 9 + (3 - 9) = (x + 3)^2 - 6, \]

so that

\[ \int \frac{1}{x^2 + 6x + 3} \, dx = \int \frac{1}{(x + 3)^2 - 6} \, dx, \]

which can be solved using the trig substitution

\[ x + 3 = \sqrt{6} \sec \theta \]

\[ dx = \sqrt{6} \sec \theta \tan \theta \, d\theta \]

The rest is left as an exercise for the reader.
Here are some more exercises:

1. \[ \int \frac{1}{x^2 + 4x + 5} \, dx \]
2. \[ \int \frac{1}{4x^2 + 4x - 3} \, dx \]
3. \[ \int \sqrt{x^2 + 2x + 2} \, dx \]
4. \[ \int \frac{1}{\sqrt{2x - x^2}} \, dx \]
5. \[ \int \frac{8x}{4x^2 + 4x - 3} \, dx \]
6. \[ \int x \sqrt{x^2 + 2x + 2} \, dx \]
Here are some solutions:

1. Complete the square: \( x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x + 2)^2 + 1 \).

Using \( x + 2 = \tan \theta \) then yields (after some additional computations)

\[
\int \frac{1}{x^2 + 4x + 5} \, dx = \arctan(x + 2) + C.
\]

2. Complete the square:

\[
4x^2 + 4x - 3 = 4(x^2 + x - \frac{3}{4}) = 4(x^2 + x + \frac{1}{4} - 1) = 4\left((x + \frac{1}{2})^2 - 1\right).
\]

Using \( x + \frac{1}{2} = \sec \theta \) then yields (after some additional computations)

\[
\int \frac{1}{4x^2 + 4x + 3} \, dx = -\frac{1}{4} \ln \left| \frac{x + \frac{1}{2}}{\sqrt{x^2 + x - \frac{3}{4}}} + \frac{1}{\sqrt{x^2 + x - \frac{3}{4}}} \right| + C.
\]

3. Complete the square: \( x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x + 1)^2 + 1 \).

Using \( x + 1 = \tan \theta \) then yields (after some additional computations)

\[
\int \sqrt{x^2 + 2x + 2} \, dx = \frac{1}{2}(x + 1)\sqrt{x^2 + 2x + 2} + \frac{1}{2} \ln |x + 1 + \sqrt{x^2 + 2x + 2}| + C.
\]

4. Complete the square:

\[
2x - x^2 = (-1)(x^2 - 2x) = (-1)(x^2 - 2x + 1) + 1 = 1 - (x - 1)^2.
\]

Using \( x - 1 = \sin \theta \) then yields (after some additional computations)

\[
\int \frac{1}{\sqrt{2x - x^2}} \, dx = \arcsin(x - 1) + C.
\]
5. Since \( \frac{d}{dx}(4x^2 + 4x - 3) = 8x + 4 \), write

\[
\int \frac{8x}{4x^2 + 4x - 3} \, dx = \int \frac{8x + 4 - 4}{4x^2 + 4x - 3} \, dx
\]

\[
= \int \frac{8x + 4}{4x^2 + 4x - 3} \, dx - 4 \int \frac{1}{4x^2 + 4x - 3} \, dx.
\]

The first integral can be finished using the substitution \( u = 4x^2 + 4x - 3 \).

The second integral was done in problem 2 above.

6. Since \( \frac{d}{dx}(x^2 + 2x) = 2x + 2 \), write

\[
\int x\sqrt{x^2 + 2x + 2} \, dx = \frac{1}{2} \int (2x + 2 - 2)\sqrt{x^2 + 2x + 2} \, dx
\]

\[
= \frac{1}{2} \int (2x + 2)\sqrt{x^2 + 2x + 2} \, dx - \int \sqrt{x^2 + 2x + 2} \, dx
\]

The first integral can be finished using the substitution \( u = x^2 + 2x + 2 \).

The second integral was done in problem 3 above.