Passing from generating functions to recursion relations

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Comments and corrections are welcome.

In the textbook you are given a method for finding the generating function (and, sometimes, a closed formula) for a sequence of real numbers \( a_n \), given a linear recursion relation for \( a_n \). This note addresses the reverse question: given the generating function for \( a_n \), can we find a linear recursion relation for \( a_n \)? When the generating function is a rational function (a ratio of polynomials), the answer is yes. Moreover, the passage from generating functions to recursions is often quite easy.

We will focus on generating functions that can be expressed as rational functions, that is, sequences \( a_n \) such that

\[
a_0 + a_1 x + a_2 x^2 + \cdots = \frac{p(x)}{q(x)},
\]

where \( p(x) \) and \( q(x) \) are polynomials in \( x \).

Here is the main result.\(^1\)

**Theorem 1.** Suppose that

\[
a_0 + a_1 x + a_2 x^2 + \cdots = \frac{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0}{c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0},
\]

where \( c_m, b_k \neq 0 \).

If \( m > k \) then

\[
c_0 a_n + c_1 a_{n-1} + \cdots + c_{m-1} a_{n-m+1} + c_m a_{n-m} = 0,
\]

for all \( n \geq m \).

If \( k \geq m \) then

\[
c_0 a_n + c_1 a_{n-1} + \cdots + c_{m-1} a_{n-m+1} + c_m a_{n-m} = 0,
\]

for all \( n > k \).

Let’s postpone the proof of this theorem for the moment and consider how it is used to find recursion relations.

\(^1\)This theorem is a special case of Theorem 4.1.1 on page 202 of *Enumerative Combinatorics, Volume 1*, by Richard Stanley.
Example 1
Suppose that
\[ a_0 + a_1 x + a_2 x^2 + \cdots = \frac{7x^2 - x - 2}{4x^3 + 3x^2 + 2x - 1}, \]
The theorem asserts that
\[ -a_n + 2a_{n-1} + 3a_{n-2} + 4a_{n-3} = 0, \]
so that
\[ a_n = 2a_{n-1} + 3a_{n-2} + 4a_{n-3}, \]
for \( n \geq 3 \). Since this recursion has order 3, we still need the three initial conditions, but these are just the first three coefficients in the generating function we were given; that is, \( a_0, a_1, a_2 \).
These initial conditions can also be obtained from the closed form of \( f(x) \). In the example just given,
\[ f(x) = \frac{7x^2 - x - 2}{4x^3 + 3x^2 + 2x - 1} = a_0 + a_1 x + a_2 x^2 + \cdots, \]
so that \( a_0 = f(0) = 2 \). More generally, we can clear the denominator to obtain
\[ 7x^2 - x - 2 = (4x^3 + 3x^2 + 2x - 1)(a_0 + a_1 x + a_2 x^2 + \cdots) \]
\[ = -a_0 + (2a_0 - a_1)x + (3a_0 + 2a_1 - a_2)x^2 + \cdots \]
Comparing coefficients of \( 1, x, x^2 \) we find that
\[ -a_0 = -2 \]
\[ 2a_0 - a_1 = -1 \]
\[ 3a_0 + 2a_1 - a_2 = 7 \]
so that
\[ a_0 = 2, \ a_1 = 5, \ a_2 = 9. \]
Now that we know \( (a_0, a_1, a_2) = (2, 5, 9) \) our recursion will do the rest. That is,
\[ a_3 = 2a_2 + 3a_1 + 4a_0 = 2(9) + 3(5) + 4(2) = 41 \]
\[ a_4 = 2a_3 + 3a_2 + 4a_1 = 2(41) + 3(9) + 4(5) = 129 \]
\[ a_5 = 2a_4 + 3a_3 + 4a_2 = 2(129) + 3(41) + 4(9) = 417 \]
\[ \vdots \]
as far as we need.
As you can see from the example, the recursion relation depends solely on the polynomial \( q(x) \) in the denominator of our generating function \( f(x) \). The numerator \( p(x) \) affects only the initial conditions.

**Example 2**

Suppose that

\[
a_0 + a_1 x + a_2 x^2 + \cdots = \frac{4x^3 - 6x^2 - 7x + 1}{2x^2 - 3x - 1},
\]

(1)

Note that, in this example, the numerator has degree 3, while the denominator has degree only 2. The theorem now asserts that, for \( n \geq 4 \), we have

\[-a_n - 3a_{n-1} + 2a_{n-2} = 0,
\]

so that

\[a_n = -3a_{n-1} + 2a_{n-2}.
\]

The recursion only holds for for \( n \geq 4 \), since the degree of the numerator is 3, while the degree of the denominator is only 2. Therefore, we still need the four initial values; that is, \( a_0, a_1, a_2, a_3 \).

To compute the initial values, we re-write Equation (1) in the form

\[
(a_0 + a_1 x + a_2 x^2 + \cdots)(2x^2 - 3x - 1) = 4x^3 - 6x^2 - 7x + 1,
\]

and compare the coefficients of \( 1, x, x^2, x^3 \) on each side. The result is

\[-a_0 = 1
\]

\[-a_1 - 3a_0 = -7
\]

\[-a_2 - 3a_1 + 2a_0 = -6
\]

\[-a_3 - 3a_2 + 2a_1 = 4
\]

so that

\[a_0 = -1, \quad a_1 = 10, \quad a_2 = -26, \quad a_3 = 94.
\]

For \( n \geq 4 \), our recursion formula \( a_n = -3a_{n-1} + 2a_{n-2} \) now yields

\[a_4 = -3(94) + 2(-26) = -335
\]

\[a_5 = -3(-335) + 2(94) = 1193
\]

and so on.
Example 3
Suppose that
\[ a_0 + a_1 x + a_2 x^2 + \cdots = \frac{x^2 + 1}{3x^3 - 2x^2 + 4x - 1}, \]
The theorem asserts that
\[ -a_n + 4a_{n-1} - 2a_{n-2} + 3a_{n-3} = 0, \]
so that
\[ a_n = 4a_{n-1} - 2a_{n-2} + 3a_{n-3}, \]
for \( n \geq 3 \), since the order of the denominator is 3 while the numerator has degree only 2. Since this is a third order recursion, we still need the three initial conditions; that is, \( a_0, a_1, a_2 \).

In this example we have
\[ \frac{x^2 + 1}{3x^3 - 2x^2 + 4x - 1} = a_0 + a_1 x + a_2 x^2 + \cdots, \]
so that
\[ x^2 + 1 = (3x^3 - 2x^2 + 4x - 1)(a_0 + a_1 x + a_2 x^2 + \cdots) \]
\[ = -a_0 + (4a_0 - a_1)x + (-2a_0 + 4a_1 - a_2)x^2 + \cdots \]
Comparing coefficients of 1, \( x \), \( x^2 \) we find that
\[ -a_0 = 1 \]
\[ 4a_0 - a_1 = 0 \]
\[ -2a_0 + 4a_1 - a_2 = 1 \]
so that
\[ a_0 = -1, \ a_1 = -4, \ a_2 = -15. \]

Now that we know \( (a_0, a_1, a_2) = (-1, -4, -15) \) our recursion will do the rest. That is,
\[ a_3 = 4a_2 - 2a_1 + 3a_0 = 4(-15) - 2(-4) + 3(-1) = -55 \]
\[ a_4 = 4a_3 - 2a_2 + 3a_1 = 4(-55) - 2(-15) + 3(-4) = -202 \]
\[ a_5 = 4a_4 - 2a_3 + 3a_2 = 4(-202) - 2(-55) + 3(-15) = -743 \]
\[ \vdots \]
and so on, far as we need.
Finally, let’s prove all of this really works.
Proof of Theorem 1. Suppose that
\[ a_0 + a_1 x + a_2 x^2 + \cdots = \frac{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0}{c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0}, \]
where \( c_m, b_k \neq 0 \).

After multiplying both sides by the denominator, we have
\[ (a_0 + a_1 x + a_2 x^2 + \cdots)(c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0. \]

Using the product formula for formal power series to simplify the left side, we then have
\[ \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} c_i a_{n-i} \right) x^n = \sum_{n=0}^{k} b_n x^n \]
Notice that the coefficient of \( x^n \) on the right side of this equation is zero when \( n > k \). It follows that
\[ \left( \sum_{i=0}^{n} c_i a_{n-i} \right) = 0 \]
for \( n > k \). Provided that \( n \geq m \) as well, this identity becomes
\[ c_0 a_n + c_1 a_{n-1} + \cdots + c_{n-1} a_1 + c_n a_0 = 0. \]

Here is a slightly more messy example.

Example 4
Suppose that
\[ a_0 + a_1 x + a_2 x^2 + \cdots = \frac{x - 3}{4 x^2 - 5 x + 2}. \]
The theorem asserts that
\[ 2a_n - 5a_{n-1} + 4a_{n-2} = 0, \]
so that
\[ a_n = \frac{5}{2} a_{n-1} - 2a_{n-2}, \quad (2) \]
for \( n \geq 2 \). Since this is a recursion of order 2, we still need two initial conditions; that is, \( a_0, a_1 \).

As before, we can clear the denominator to obtain
\[ x - 3 = (4 x^2 - 5 x + 2)(a_0 + a_1 x + a_2 x^2 + \cdots) \]
\[ = 2a_0 + (2a_1 - 5a_0)x + \cdots \]
Comparing coefficients of $1, x$, we find that
\[
2a_0 = -3 \\
2a_1 - 5a_0 = 1
\]
so that
\[
a_0 = -\frac{3}{2}, \quad a_1 = -\frac{13}{4}.
\]
Now that we know $(a_0, a_1) = (-\frac{3}{2}, -\frac{13}{4})$, the recursion relation (2) then yields
\[
a_2 = \frac{5}{2}(-\frac{13}{4}) - 2(-\frac{3}{2}) = -\frac{41}{8}
\]
\[
a_3 = \frac{5}{2}(-\frac{41}{8}) - 2(-\frac{13}{4}) = -\frac{101}{16}
\]
and so on.

We conclude with an example that demonstrates how the methods used above can also be applied to generating functions of other forms.

**Example 5**
Suppose that
\[
a_0 + a_1 x + a_2 x^2 + \cdots = \frac{e^x}{1 - x - x^2}.
\]
Since this is not a rational function (ratio of polynomials) we cannot apply Theorem 1. However, we know from elementary calculus that
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]
so we can follow the same approach as earlier, clearing the denominator to obtain
\[
\sum_{n=0}^{\infty} x^n = (a_0 + a_1 x + a_2 x^2 + \cdots)(1 - x - x^2)
\]
\[
= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots
\]
so that
\[
1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} = a_0 + (a_1 - a_0)x + \sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2})x^n.
\]
Comparing coefficients on each side yields

\[
\begin{align*}
    a_0 &= 1 \\
    a_1 - a_0 &= 1 \\
    a_n - a_{n-1} - a_{n-2} &= \frac{1}{n!} \quad \text{for } n \geq 2,
\end{align*}
\]

so that

\[
\begin{align*}
    a_0 &= 1 \\
    a_1 &= 2 \\
    a_n &= a_{n-1} + a_{n-2} + \frac{1}{n!} \quad \text{for } n \geq 2.
\end{align*}
\]

References and recommended further reading

