1. How to compute the orthogonal matrix that represents a rotation of $\mathbb{R}^3$

Recall that the $2 \times 2$ matrix

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

rotates the plane $\mathbb{R}^2$ counter-clockwise by the angle $\theta$ around the origin. Is there a similar way to represent rotations of 3-dimensional space using $3 \times 3$ matrices?

Consider the simple case of rotating 3-dimensional space by the same angle $\theta$ counter-clockwise around the $z$-axis. This is analogous to rotating the earth by the angle $\theta$ around the north pole, for example. This rotation fixes the $z$-axis, and acts on the $xy$-plane in the exactly the same way as the $2 \times 2$ matrix $A_\theta$ above. Therefore, the matrix of rotation around the $z$-axis by the counter-clockwise angle $\theta$ is given by

$$S_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where we assign the column vector $(1, 0, 0)^T$ to the $x$-axis, $(0, 1, 0)^T$ to the $y$-axis, and $(0, 0, 1)^T$ to the $z$-axis.

Note that, like $A_\theta$, the matrix $S_\theta$ is an orthogonal matrix, that is,

$$S_\theta S_\theta^T = I \quad \text{or, equivalently,} \quad S_\theta^T = S_\theta^{-1}.$$ 

More generally, suppose we rotate 3-dimensional space counter-clockwise by the angle $\theta$ around a different axis through the origin, pointing along the direction of some unit vector $u \in \mathbb{R}^3$. For this we need the analogue of the matrix $S_\theta$, for which the $z$-axis is replaced by a different axis of rotation, the line passing through the point $u$ and the origin $o$. Let us call this new rotation matrix $R_{\theta,u}$, depending as it does on both the choice of axis $u$ and the angle of rotation $\theta$.

To compute $R_{\theta,u}$, choose a unit vector $v$ that is orthogonal to $u$; that is, so that $u \cdot v = 0$. Let $w = u \times v$, where $\times$ denotes the vector cross product in $\mathbb{R}^3$. We now have a new orthonormal basis for $\mathbb{R}^3$, $\{v, w, u\}$ such that $v \times w = u$. (It might help the reader to sketch this basis, where $u$ is the vector pointing upwards in your picture, in analogy to the $z$-axis.)

Let $P$ denote the matrix having $v$, $w$, $u$ as its three columns (in that exact order):

$$P = \begin{bmatrix} v & w & u \end{bmatrix}, \quad (1)$$
Note that $P$ is an orthogonal matrix, $P^T P = I$, since the columns of $P$ were (deliberately) chosen to form an orthonormal set. Note in particular that
\[
P^T u = \begin{bmatrix} v^T \\ w^T \\ u^T \end{bmatrix} u = \begin{bmatrix} v^T u \\ w^T u \\ u^T u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\]
and that, similarly,
\[
P^T v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad P^T w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

**Theorem 1** The matrix $R_{\theta, u}$ that rotates $\mathbb{R}^3$ around the vector $u$ by the counterclockwise angle $\theta$ is given by the formula
\[
R_{\theta, u} = PS_{\theta} P^T
\]

**Proof of Theorem 1:** To begin, consider what the transformation $PS_{\theta} P^T$ does to the vectors $v, w, u$. The matrix $PS_{\theta} P^T$ fixes $u$; indeed, by (1) and (2),
\[
PS_{\theta} P^T u = PS_{\theta} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = u.
\]
Similarly,
\[
PS_{\theta} P^T v = PS_{\theta} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} = \cos \theta v + \sin \theta w,
\]
while
\[
PS_{\theta} P^T w = PS_{\theta} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix} = -\sin \theta v + \cos \theta w.
\]
More generally, if $X = av + bw + cu$ is any vector in $\mathbb{R}^3$ (expressed in terms of the orthonormal basis $\{v, w, u\}$) then
\[
PS_{\theta} P^T X = aPS_{\theta} P^T v + bPS_{\theta} P^T w + cPS_{\theta} P^T u
\]
\[
= a(\cos \theta v + \sin \theta w) + b(-\sin \theta v + \cos \theta w) + cu = R_{\theta, u} X,
\]
rotating $X$ counterclockwise by $\theta$ in the $vw$-plane orthogonal to the axis of rotation $u$. 

The identity (3), together with the orthogonality of $P$ and $S_{\theta}$, implies that $R_{\theta, u}$ is also an orthogonal matrix. More precisely, we have the following corollary.

**Corollary 2** A rotation matrix $R$ is an orthogonal matrix with determinant 1.
Proof: If $R$ is a rotation matrix then $R = P S_\theta P^T$, where $P^T = P^{-1}$ and $S_\theta^T = S_\theta^{-1}$, as in Theorem 1. Therefore,

$$R^T R = (P S_\theta P^T)^T P S_\theta P^T = P S_\theta^T P^T P S_\theta P^T = P S_\theta^T S_\theta P^T = PP^T = I,$$

so that $R$ is an orthogonal matrix. Moreover,

$$\det(R) = \det(P S_\theta P^T) = \det(P S_\theta P^{-1}) = \det(P) \det(S_\theta) \frac{1}{\det(P)} = \det(S_\theta) = 1.$$

\[\square\]

Remark: The converse of the Corollary is also true: A matrix $R$ is a rotation matrix if and only if $R$ is an orthogonal matrix and $\det(R) = 1$. But we will not prove this now.

Example: Find the matrix $R_{\pi/3, \mathbf{u}}$ that rotates $\mathbb{R}^3$ by the counterclockwise angle $\pi/3$ around the axis through the vector $\mathbf{u} = (2, 1, 1)$.

Solution: To begin, find a vector $\mathbf{v}$ that is perpendicular to $\mathbf{u} = (2, 1, 1)$. An easy choice is $\mathbf{v} = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. We then set $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, so that

$$\mathbf{w} = \det\begin{bmatrix} i & j & k \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = (-2, 2, 2).$$

We now have an orthogonal set $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$, where $\mathbf{v} \times \mathbf{w} = \mathbf{u}$ and $\mathbf{u}$ is parallel to our desired axis of rotation. Unfortunately, however, the vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$ are not unit vectors. This is easily fixed: dividing each vector by its length, re-assign the variables $\mathbf{v}, \mathbf{w}, \mathbf{u}$ to form the orthonormal set:

$$\mathbf{v} = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \quad \mathbf{w} = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \quad \mathbf{u} = (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),$$

so that

$$P = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

It now follows from Theorem 1 that

$$R_{\pi/3, \mathbf{u}} = PS_{\pi/3} P^T = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\sqrt{3}/2 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

After multiplying these matrices, we obtain

$$R_{\pi/3, \mathbf{u}} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{6} + \frac{1}{2\sqrt{2}} \\ \frac{1}{6} + \frac{1}{2\sqrt{2}} & \frac{7}{12} & \frac{1}{12} - \frac{1}{\sqrt{2}} \\ \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{12} + \frac{1}{\sqrt{2}} & \frac{7}{12} \end{bmatrix}.$$

(4)
2. How to compute the rotation of $\mathbb{R}^3$ represented by a given orthogonal matrix

Now suppose you are given an orthogonal matrix $R$ such that $\det R = 1$; in other words, a rotation matrix. What is the axis of rotation for $R$? What is the angle of rotation? How do we compute $u$ and $\theta$ so that $R = R_{\theta,u}$?

Here is one quick test to find $\theta$. Recall that the trace of a square $n \times n$ matrix $A$ is the sum of its diagonal entries: $\text{trace}(A) = A_{11} + A_{22} + \cdots + A_{nn}$.

**Theorem 3 (The Cosine Test)** If $R$ is a rotation matrix having angle of rotation $\theta$, then

$$\cos \theta = \frac{\text{trace}(R) - 1}{2}. \quad (5)$$

**Proof:** We will need the fact that if $A$ is any square $n \times n$ matrix, and $P$ is an $n \times n$ invertible matrix, then $\text{trace}(PAP^{-1}) = \text{trace}(A)$. This is a consequence of the fact that, for any two $n \times n$ matrices $A$ and $B$, we have $\text{trace}(AB) = \text{trace}(BA)$. (You can check this directly by using the matrix multiplication formula.)

If $R$ is a rotation matrix having angle of rotation $\theta$, then $R = R_{\theta,u}$ for some unit vector $u$, so that $R = PS_{\theta}P^T = PS_{\theta}P^{-1}$, as in (3). Hence,

$$\text{trace}(R) = \text{trace}(PS_{\theta}P^{-1}) = \text{trace}(S_{\theta}) = 1 + 2 \cos \theta,$$

from which the formula (5) above immediately follows.  \[\blacksquare\]

The Cosine Test, while very easy to use, doesn’t tell the whole story, since the axis of rotation $u$ remains unknown. Moreover, there remains an ambiguity regarding the value of $\theta$, since we only know $\cos \theta$. Since $\cos \theta = \cos(-\theta)$, the sign of the angle remains obscure.

Fortunately it takes only a tiny bit of work to compute $u$. The key is to remember that if $u$ lies in the axis of rotation, then the rotation $R$ fixes the vector $u$. In other words, $Ru = u$. Since the inverse matrix $R^{-1}$ will represent rotation around the same axis $u$ by the negative of the angle $\theta$, we also have $R^{-1}u = u$. Recall that $R$ is an orthogonal matrix, so that $R^T = R^{-1}$. It now follows that $R^T u = R^{-1} u = u$, so that

$$(R - R^T)u = Ru - R^T u = u - u = 0.$$ 

This suggests that we can discover the vector $u$ by considering the null space of the matrix $R - R^T$.

Denote

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$
We then have

\[
R - R^T = \begin{bmatrix}
0 & r_{12} - r_{21} & r_{13} - r_{31} \\
r_{21} - r_{12} & 0 & r_{23} - r_{32} \\
r_{31} - r_{13} & r_{32} - r_{23} & 0
\end{bmatrix} = \begin{bmatrix}
0 & \alpha & \beta \\
-\alpha & 0 & \gamma \\
-\beta & -\gamma & 0
\end{bmatrix},
\]

where we denote \( \alpha = r_{12} - r_{21} \), \( \beta = r_{13} - r_{31} \), and \( \gamma = r_{23} - r_{32} \). This suggests that \( u \) is parallel to the vector

\[
q = \begin{bmatrix}
-\gamma \\
\beta \\
-\alpha
\end{bmatrix} = \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12}
\end{bmatrix}, \tag{6}
\]

assuming this vector is not the zero vector (which might happen sometimes).

We summarize this result, and make it more precise, with the following theorem.

**Theorem 4 (The Symmetric Difference Test)** Suppose that \( R \) is a rotation matrix, and suppose that \( R^T \neq R \), so that the vector \( q \neq 0 \). Then the axis of rotation of \( R \) is parallel to \( q \). More specifically, the matrix \( R \) rotates \( \mathbb{R}^3 \) by a positive counterclockwise angle \( \theta \) around the unit vector \( u \), where

\[
q = 2(\sin \theta) u.
\]

Note, in particular, that \( 2 \sin \theta = |q| \) and \( u = \frac{q}{|q|} \). Using both Theorem 3 and Theorem 4 we obtain the axis of rotation, with direction and orientation provided by \( u \), and the exact value of the angle \( \theta \), from the values of \( \cos \theta \) and \( \sin \theta \).

**Proof:** Suppose that \( R = R_{\theta,u} = PS_\theta P^T \) as in (3). Then

\[
\begin{bmatrix}
0 & \alpha & \beta \\
-\alpha & 0 & \gamma \\
-\beta & -\gamma & 0
\end{bmatrix} = R - R^T = PS_\theta P^T - PS_\theta^T P^T = P(S_\theta - S_\theta^T) P^T = P \begin{bmatrix}
0 & -2\sin \theta & 0 \\
2\sin \theta & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} P^T
\]

\[
= \begin{bmatrix}
v & w & u
\end{bmatrix} \begin{bmatrix}
0 & -2\sin \theta & 0 \\
2\sin \theta & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
v^T \\
w^T \\
u^T
\end{bmatrix} = 2(\sin \theta)(wv^T - vw^T),
\]

so that

\[
\alpha = 2 \sin \theta(v_2w_1 - v_1w_2), \quad \beta = 2 \sin \theta(v_3w_1 - v_1w_3), \quad \gamma = 2 \sin \theta(v_3w_2 - v_2w_3).
\]

In other words,

\[
q = \begin{bmatrix}
-\gamma \\
\beta \\
-\alpha
\end{bmatrix} = 2(\sin \theta) v \times w = 2(\sin \theta) u.
\]

\[\square\]
**Example:** Let’s use Theorem 4 to check the work we did in the last example, where

\[
R = \begin{bmatrix}
\frac{5}{6} & \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{6} + \frac{1}{2\sqrt{2}} \\
\frac{1}{6} + \frac{1}{2\sqrt{2}} & \frac{7}{12} & \frac{1}{12} - \frac{1}{\sqrt{2}} \\
\frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{12} + \frac{1}{\sqrt{2}} & \frac{7}{12}
\end{bmatrix}.
\]

In this case, we use (6) to compute

\[
q = \begin{bmatrix}
\frac{2}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix},
\]

so that \(2\sin \theta = |q| = \sqrt{3}\). This implies that \(\theta = \arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}\), and that \(u\) is parallel to \((2, 1, 1)\), as we began with in the previous example.

We can double-check the angle calculation with the Cosine Test. In this case, we have

\[
\cos \theta = \frac{\text{trace}(R) - 1}{2} = \frac{1}{2} (\frac{5}{6} + \frac{7}{12} + \frac{7}{12} - 1) = \frac{1}{2},
\]

so that \(\theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}\) once again.

**Question:** Theorem 4 assumes that \(R \neq R^T\). What if \(R = R^T\)? In this case we get \(R - R^T = 0\), the zero matrix, so that \(q = 0\), the zero vector. From this we can deduce that \(\sin \theta = 0\), so that either \(\theta = 0\) or \(\theta = \pi\). If \(\theta = 0\), then \(R\) is the identity rotation, and this would be obvious immediately, since \(R\) would be the identity matrix! So if \(R \neq I\) we know that \(\theta = \pi\). But what is the axis of rotation? Since \(\theta = \pi\) in this instance, we have

\[
R = PS_\pi P^T = P \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} P^T = -vv^T - ww^T + uu^T.
\]

Since \(v, w, u\) form an orthonormal basis, \(vv^T + ww^T + uu^T = I\), the identity matrix (Why?), so that

\[
R = -vv^T - ww^T - uu^T + 2uu^T = -I + 2uu^T,
\]

and \(2uu^T = I + R\). But the columns of the matrix \(uu^T\) are each parallel to \(u\) (Why?), so the vector \(u\) can be obtained by taking any non-zero column of \(I + R\) and normalizing to a unit vector.
3. Summary

To compute $R_{\theta,u}$ from a unit vector $u$ and an angle $\theta$:

(1) Choose any unit vector $v$ such that $v \perp u$.

(2) Set $w = u \times v$ and set $P = \begin{bmatrix} v & w & u \end{bmatrix}$.

(3) The matrix $R_{\theta,u}$ is given by

$$R_{\theta,u} = P S_\theta P^T = \begin{bmatrix} v & w & u \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^T \\ w^T \\ u^T \end{bmatrix}.$$ 

To compute $u$ and $\theta$ from a rotation matrix $R$:

(1) If $R \neq R^T$, then set $q = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$.

In this case $R = R_{\theta,u}$ where $u = q/|q|$ and $\sin \theta = |q|/2$, and $\cos \theta = \frac{\text{trace}(R) - 1}{2}$.

(2) If $R = R^T$ and $R \neq I$ then $R = R_{u,\pi}$ where $u$ is a unit vector parallel to any non-zero column of $I + R$.

(3) If $R = I$ then $R$ is the identity rotation (angle zero, everything stays fixed).
Exercises:

1. Compute the matrix $R_{\frac{\pi}{4}, (1, 1, 1)}$.

2. Compute the matrix $R_{\frac{\pi}{6}, (0, 1, 0)}$.

3. Compute the matrix $R_{\pi, (2, 0, 1)}$.

4. Compute the matrix $R_{2\pi, (2, 0, 1)}$.

5. Compute the angle $\theta$ and axis of rotation $\mathbf{u}$ for the rotation matrix

\[
R = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}
\end{bmatrix}.
\]

6. Compute the angle $\theta$ and axis of rotation $\mathbf{u}$ for the rotation matrix

\[
R = \begin{bmatrix}
\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\
\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\
0 & \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}}
\end{bmatrix}.
\]

7. Compute the angle $\theta$ and axis of rotation $\mathbf{u}$ for the rotation matrix

\[
R = \begin{bmatrix}
-\frac{2}{3} & -\frac{3}{3} & -\frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{2}{3}
\end{bmatrix}.
\]

8. Suppose that $R = R_{\theta, \mathbf{u}}$. Prove that $R^{-1} = R_{-\theta, \mathbf{u}}$.

9. Suppose that $R = R_{\theta, \mathbf{u}}$. Prove that $R^{T} = R_{-\theta, \mathbf{u}}$.

10. Suppose that $R = R_{\theta, \mathbf{u}}$. Prove that $R^{2} = R_{2\theta, \mathbf{u}}$.

11. Prove that $R_{-\theta, \mathbf{u}} = R_{\theta, -\mathbf{u}}$.

12. Suppose that $R = R_{\pi, \mathbf{u}}$. Prove that $(I + R)\mathbf{v} = 0$ and that $(I + R)\mathbf{w} = 0$. 
Selected Solutions:

1. $R_{\pi,(1,1,1)} = \begin{bmatrix}
\frac{1}{3} + \frac{2}{3\sqrt{2}} & \frac{1}{3} - \frac{1}{\sqrt{6}} & \frac{1}{3} + \frac{1}{\sqrt{6}} \\
\frac{1}{3} - \frac{1}{\sqrt{6}} & \frac{1}{3} + \frac{2}{3\sqrt{2}} & \frac{1}{3} - \frac{1}{\sqrt{6}} \\
\frac{1}{3} + \frac{2}{3\sqrt{2}} & \frac{1}{3} - \frac{1}{\sqrt{6}} & \frac{1}{3} + \frac{2}{3\sqrt{2}}
\end{bmatrix}$.

2. $R_{\pi,(0,1,0)} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}
\end{bmatrix}$.

3. $R_{\pi,(2,0,1)} = \begin{bmatrix}
0 & 1 & 0 \\
\frac{4}{5} & 0 & -\frac{3}{5}
\end{bmatrix}$.

4. $R_{2\pi,(2,0,1)} = I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$.

5. $q = \begin{bmatrix}
-\sqrt{2} \\
\frac{\sqrt{3} - 1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}}
\end{bmatrix}$, and $R = R_{u,\theta}$, where $u = \frac{q}{|q|} \approx \begin{bmatrix}
-0.743 \\
-0.594 \\
-0.308
\end{bmatrix}$

and $\theta = \arcsin\left(\frac{|q|}{2}\right) = \arccos\left(\text{trace}(R) - 1\right) \approx 1.217$ radians.

6. $q = \begin{bmatrix}
\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{30}} \\
\frac{2}{\sqrt{30}} - \frac{2}{\sqrt{6}} \\
\frac{\sqrt{2}}{\sqrt{5}} - \frac{1}{\sqrt{15}}
\end{bmatrix}$, and $R = R_{u,\theta}$, where $u = \frac{q}{|q|} \approx \begin{bmatrix}
0.845 \\
0.522 \\
0.111
\end{bmatrix}$

and $\theta = \arcsin\left(\frac{|q|}{2}\right) = \arccos\left(\text{trace}(R) - 1\right) \approx 2.785$ radians.

7. Since $R^T = R$ and $R \neq I$, it follows that $\theta = \pi$. We then compute

$$I + R = \begin{bmatrix}
\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\
-\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3}
\end{bmatrix},$$

so that $u = \begin{bmatrix}
\frac{1}{\sqrt{6}} \\
\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{6}}
\end{bmatrix}$ and $R = R_{\pi,(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}})}$.  
