92.490/651 Supplemental Exercises **due in class on 4/26/2010**

To help you with some of these exercises I have posted some extra online notes on volume and determinants that elaborate some of the ideas discussed in lecture. Those notes can be found at: `faculty.uml.edu/dklain/volume-formula.pdf`

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1. Find the area V_2 of the parallelogram in $\mathbb{R}^2$ having vertices at the four points:

$$(2,1) \quad (4,5) \quad (7,3) \quad (5,-1)$$

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2. Find the area V_2 of the parallelogram in $\mathbb{R}^5$ having vertices at the four points:

$$(2,1,1,1,1) \quad (4,5,0,1,2) \quad (7,3,0,4,2) \quad (5,-1,1,4,1)$$

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3. Let τ_n denote the n-volume of the unit equilateral n-simplex T_n. For example, $\tau_2 = \frac{\sqrt{3}}{4}$, the area of a unit equilateral triangle, while τ_3 is the volume of the tetrahedron having all unit length edges.

The following steps lead to a formula for τ_n.

Let $e_1 = (1,0,0\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, ..., $e_{n+1} = (0,0,0,\ldots,1)$, denote the standard coordinate unit vectors in $\mathbb{R}^{n+1}$. The convex hull S of these points is an equilateral n-simplex lying inside the plane $x_1 + \cdots + x_{n+1} = 1$ of $\mathbb{R}^{n+1}$. Note, however, that it is not the *unit* equilateral simplex, since each edge of S has length $\sqrt{2}$.

(a) Show that the $(n+1)$-dimensional volume of the $(n+1)$-dimensional simplex S' having vertices at $o, e_1, \ldots, e_{n+1}$ is

$$V_{n+1}(S') = \frac{1}{(n+1)!}.$$

Hint: Induction on dimension might be helpful.

(b) Now use the cone volume formula in a different way to find the n-volume of S.
Hint: Think of S' as the convex hull of S with the origin.

(c) Now tweak the answer to part (b) to get a formula for τ_n.

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4. Let ω_n denote the volume V_n of the n-dimensional ball having unit radius. Use an elementary slicing and integration argument to show that

$$\omega_n = 2\omega_{n-1} \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$$

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5. For $t > 0$ denote by $\Gamma(t)$ the Gamma function, given by

$$\Gamma(t) = \int_0^{\infty} \mathrm{e}^{-x} x^{t-1} \mathrm{d}x.$$

(a) Prove that $\Gamma(1) = 1$.

(b) Prove that $\Gamma(t+1) = t\Gamma(t)$. (*Hint:* Try integration by parts.)

(c) Prove that if n is a positive integer then $\Gamma(n+1) = n!$.

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6. This exercise will show you how to compute $\Gamma\left(\frac{1}{2}\right)$, a value to be used later for computing the volume the unit ball.

(a) Use a simple substitution to show that

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \, dx$$

(b) Now use part (a) to show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Here are some hints: We can think of the square of the integral from part (a) as the product of integrals in two independent variables, x and y.

$$\left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \, \mathrm{d}x\right)^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \, \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} \, \mathrm{d}y = \int_{-\infty}^{\infty} \mathrm{e}^{-(x^2+y^2)} \, \mathrm{d}x \, \mathrm{d}y$$

Now convert from xy-coordinates to polar coordinates (r, θ) and finish the computation.

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7. In this exercise we find a formula for the volume ω_n of the unit n-ball. I'll start it for you. Following a similar argument to the previous problem, but using n-variables, we have

$$\begin{aligned}
\pi^{n/2} &= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)^n \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-x_1^2}\,\mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{e}^{-x_n^2}\,\mathrm{d}x_n \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-(x_1^2+\cdots+x_n^2)}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \int_{u\in\mathbb{S}^{n-1}} \int_0^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r\,\mathrm{d}u \\
&= n\omega_n \int_0^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r,
\end{aligned}$$

since the surface area of the sphere $\mathbb{S}^{n-1}$ is equal to $n\omega_n$. (Notice the change of variables from Cartesian to spherical coordinates in $\mathbb{R}^n$.)

Now *you* finish the computation: substituting $y = r^2$, compute the integral

$$\int_0^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r,$$

and solve for ω_n in terms of the gamma function.

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8. Let n be a non-negative integer. Use the formula you proved in the previous problem to prove that

$$\omega_{2n} = \frac{\pi^n}{n!} \quad \text{and} \quad \omega_{2n+1} = \frac{2^{2n+1}\pi^n n!}{(2n+1)!}$$