Throughout this set denote by V the 3-dimensional volume and by A the area (2-dimensional volume).

1. Suppose $K_0, K_1 \in \mathscr{K}_3$ lie respectively inside parallel 2-dimensional planes H_0, H_1 of $\mathbb{R}^3$, and denote by h the orthogonal distance (height) between these hyperplanes. The convex hull $Q = conv(K_0 \cup K_1)$ is called a *prismatoid* with *bases* K_0 and K_1 and *height* h.

If K_0 and K_1 are translation congruent sets, the Q is just a prism (or cylinder) with base K_0 and height h. If K_1 is a single point, then Q is a cone. The prismatoid is a generalization of these two geometric notions.

(a) Prove that the horizontal planar slice (parallel to H_0 and H_1) at height $z \in [0, h]$ is given by the set $(1 - \frac{z}{h})K_0 + \frac{z}{h}K_1$.

(b) Use the formula for areas of Minkowski sums (and mixed areas) to prove that

$$V(Q) = \frac{h}{3}\Big(A(K_0) + A(K_0, K_1) + A(K_1)\Big).$$

Hint: Set up a volume integral for Q by integrating the areas of suitable parallel slices of Q.

❧

2. Recall that if P is a polytope in $\mathbb{R}^3$ having facet unit normals $u_1, \ldots, u_m$, then

$$V(P) = \frac{1}{3}\sum_{i=1}^{m} h_P(u)A(P^u).$$

Let P, Q be polytopes and $\epsilon > 0$. Use the previous identity and our work on mixed areas to show that $V(P + \epsilon Q)$ is a cubic polynomial in ϵ. Specifically, show that

$$V(P + \epsilon Q) = V(P) + c_1\epsilon + c_2\epsilon^2 + V(Q)\epsilon^3,$$

where c_1 and c_2 are constants that depend only on P and Q.

(Your answer should express c_1 and c_2 in terms of support functions and mixed areas of various facets of P and Q.)

❧

Remark: Using a standard continuity argument it follows that, for any $K, L \in \mathscr{K}_3$, the function $V(K + \epsilon L)$ is a cubic polynomial in ϵ as well (even when K and L are not polytopes).

❧

3. Following the notation of the previous problem, define

$$V(P, P, Q) = \tfrac{1}{3}c_1 \quad \text{and} \quad V(Q, Q, P) = \tfrac{1}{3}c_2.$$

and prove that

$$3V(P, P, Q) = \lim_{\epsilon \to 0} \frac{V(P + \epsilon Q) - V(P)}{\epsilon}.$$

Remark: Once again it follows by an approximation argument that the same result holds for arbitrary $K, L \in \mathscr{K}_3$.

❧

4. Let B denote the unit ball in $\mathbb{R}^3$ and let u be a unit vector. Prove that

$$\begin{aligned} &(\mathbf{a})\ \ V(K,K,K) = V(K) \\ &(\mathbf{b})\ \ 3V(K,K,B) = S(K) \\ &(\mathbf{c})\ \ V(K,K,u) = 0 \\ &(\mathbf{d})\ \ 3V(K,K,\overline{ou}) = A(K|u^{\perp}) \end{aligned}$$

❧

5. Let $K \in \mathscr{K}_2$, let m be a positive integer, and let $\phi_1, \ldots, \phi_m$ denote a sequence of rotations of $\mathbb{R}^2$ around the origin. The set

$$\tfrac{1}{m}\Big(\phi_1 K + \cdots + \phi_m K\Big)$$

is called a *rotation mean* of K. Let $\mathcal{M}_K$ denote the set of *all* rotation means of K.

(a) Prove that, if L is a rotation mean of K, then K and L have the same perimeter.

(b) Prove that, if L is a rotation mean of K, then $A(L) \geq A(K)$.

(c) Prove that equality holds in part (b) if and only if K and L are homothetic.

(d) Prove that there is a ball of some radius $r > 0$ such that, if L is a rotation mean of K, then $L \subseteq rB$.

Remark: It follows from part (d) and the compactness theorem that every infinite sequence of rotation means of K has a convergent subsequence. It also follows from this uniform boundedness that the *areas* of rotation means of K are bounded above by some finite number. Let

$$\alpha = \sup_{L \in \mathcal{M}_K} A(L).$$

By the compactness theorem there is a sequence $\{L_i\}$ in $\mathcal{M}_K$ that converges to a convex body L_* such that $A(L_*) = \alpha$. Note that L_* itself might not be a rotation mean of K.

(e) Prove that L_* is a disk.

(f) If $Perimeter(K) = 1$ then what is α?

❧

6. Suppose that $f : \mathscr{K}_2 \to \mathbb{R}$ is a continuous function on convex bodies such that

$$f(K + L) = f(K) + f(L)$$

for all $K, L \in \mathscr{K}_2$. Suppose also that $f(K + v) = f(K)$ and $f(\phi K) = f(K)$ for all $K \in \mathscr{K}_2$, all $v \in \mathbb{R}^2$, and every rotation ϕ. (In other words, f is translation invariant and rotation invariant.)

Let B denote the unit disc in $\mathbb{R}^2$, and let $c = f(B)$. Prove that, if $K \in \mathscr{K}_2$, then

$$f(K) = \frac{c}{2\pi} Perimeter(K).$$

Hint: Use the rotation means from problem **5** above.

Remark: This problem shows that, up to a constant multiple, perimeter is the only continuous, rigid-motion invariant, Minkowski additive function on convex sets in the plane.