Determinants and volume

These notes elaborate a topic covered in lecture on Monday, April 12.

Let $X_1, \ldots, X_k$ be a family of column vectors in $\mathbb{R}^n$, and let

$$Q = \{a_1 X_1 + \cdots + a_k X_k \mid 0 \leq a_i \leq 1\}.$$

If the vectors $X_1, \ldots, X_k$ are linearly independent, then the set Q is a k-dimensional parallelopiped in $\mathbb{R}^n$.

What is the k-dimensional volume of this parallelopiped? (By k-dimensional volume, we mean the volume of Q taken inside the k-dimensional affine hull of Q)

Set up a matrix A using the X_i as the columns of A, that is, let

$$A = \left[\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_k \end{array} \right].$$

Note that each vector $X_i \in \mathbb{R}^n$ has n coordinates, so that A is an $n \times k$ matrix. We will prove the following formula for the k-volume of a k-dimensional parallelopiped in $\mathbb{R}^n$.

Theorem 1.

$$V_k(Q) = \sqrt{\det(A^T A)}.$$

Here A^T denotes the matrix transpose of A.

Proof. Perform the Gram-Schmidt process on the columns of A to obtain a new set of columns that are *orthogonal*. This process is performed by doing certain *column operations* on A that add (or subtract) multiples of one column from another. Each operations of this kind can be accomplished by multiplying *on the right* by a square $k \times k$ matrix of determinant 1.

Meanwhile, these column operations can interpreted geometrically as slicing off one side of the parallelopiped and gluing that slice back onto the opposite end, so that k-volume is also preserved.

The end result is a new matrix AM, where the $k \times k$ invertible matrix M is product of (invertible) elementary matrices (representing the column operations we had to do on A), so that $\det(M) = 1$.

The matrix AM is a new $n \times k$ matrix

$$AM = \left[\begin{array}{c|c|c|c} Y_1 & Y_2 & \cdots & Y_k \end{array} \right],$$

such that $Y_1, \ldots, Y_k$ are orthogonal, and such that the new box

$$Q' = \{a_1 Y_1 + \cdots + a_k Y_k \mid 0 \le a_i \le 1\},$$

has the *same k-volume* as the original box Q.

Since the columns Y_i of AM are orthogonal, we have

$$(AM)^T(AM) = \begin{bmatrix} Y_1^T Y_1 & 0 & \cdots & 0 \\ 0 & Y_2^T Y_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Y_k^T Y_k \end{bmatrix} = D,$$

where D a diagonal matrix, whose i-th diagonal entry $d_i = Y_i^T Y_i = |Y_i|^2$. the square of the length of the vector Y_i.

Since Q' is a box with orthogonal edges, its k-volume is easy to compute; it's just the product of the lengths of the edges in each direction. So

$$V_k(Q') = |Y_1||Y_2| \cdots |Y_k|.$$

Hence,

$$V_k(Q')^2 = |Y_1|^2|Y_2|^2 \cdots |Y_k|^2 = \det[(AM)^T(AM)].$$

But recall that $\det(M) = 1$. It follows that

$$\det[(AM)^T(AM)] = \det[M^T(A^T A)M] = \det(M^T)\ \det(A^T A)\ \det(M) = \det(A^T A).$$

Putting everything together, we find that

$$V_k(Q)^2 = V_k(Q')^2 = \det[(AM)^T(AM)] = \det(A^T A),$$

so that

$$V_k(Q) = \sqrt{\det(A^T A)}.$$

□

For example, to compute the area V_2 of the parallelogram $Q_{\mathbf{v},\mathbf{w}}$ in $\mathbb{R}^n$ with vertices at the origin o and the points $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}+\mathbf{w}$, set A to be the $n \times 2$ matrix

$$A = \left[\begin{array}{c|c} & \\ \mathbf{v} & \mathbf{w} \\ & \end{array}\right].$$

In this case, $A^T A$ is the 2×2 matrix of vector dot products:

$$A^T A = \left[\begin{array}{c|c} \mathbf{v}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{w} \\ \hline \mathbf{w}\cdot\mathbf{v} & \mathbf{w}\cdot\mathbf{w} \end{array}\right],$$

so that

$$V_2(Q_{\mathbf{v},\mathbf{w}}) = \sqrt{(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})^2}.$$