Physics I
95.141

LECTURE 0

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Outline

- Algebra Review
- Trig Review
- Useful geometric formulas
- Coordinate systems
- Functions
- Derivatives
- Integrals
- Differential equations

Basic Algebra

- It is absolutely essential that you be comfortable with Algebra for Physics I, many of the problems we will do rely on your ability to solve algebraic expression quickly.
- For instance, if you have an expression

$$10 = 5x$$

You want to solve for x, so divide both sides of the equation by 5.

$$\frac{10}{5} = \frac{5}{5}x \longrightarrow x = 2$$

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Basic Algebra

- The last expression was easy…often, we will be faced with more complicated expressions.
- In particular, we will often have to solve quadratic equation, which are of the form:

$$ax^2 + bx + c = 0$$

Where a,b, and c are constants, and x is the variable you wish to solve for.

In this case, x has 2 solutions, given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Basic Algebra

- Often we will need to solve a “system of equations”. A system of equations is more than one equation, and more than one unknown. Such a system is solvable if you have at least as many equations as unknowns.
- For example, say you have the following two equations:

$$4x + 5y = 43$$

$$x - y = 4$$

- We can solve this in a couple of ways

Basic Algebra

- Let's try substitution
- Here we rewrite the second equation in terms of x by adding y to each side of the equation:

$$x = 4 + y$$

- We then take this expression for x, and substitute it into the first equation:

$$4x + 5y = 43$$

$$4(4 + y) + 5y = 43$$

- We now have an expression entirely in terms of y

$$16 + 4y + 5y = 43$$

$$9y = 27$$

$$y = 3$$

Now that we have y, we can plug it into either equation and get x

$$x = 4 + y \quad \text{so} \quad x = 4 + 3 = 7$$

Basic Algebra

- Now let's try subtraction
- Take equation 2 and multiple both sides by 4, this gives:

$$4(x-y)=4\times 4$$

$$4x-4y=16$$

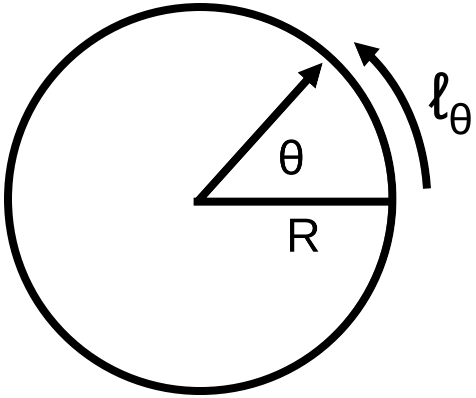
- We then take our new equation 2, and subtract it from equation 1

$$\begin{array}{r} 4x+5y=43 \\ \underline{-\ 4x-4y=16} \\ 0+9y=27 \end{array}$$

- As you can see, this also gives us y=3. We can then plug y back into either equation 1 or 2 and get x=7.
- The two methods are equavalent, and you should be proficient in both!

Trig Review

- Angles can be measured in either radians [rad] or degrees [º]
- If we think about a circle:



- Angle in Radians is the really the ratio of the length of the arc subtended to the Radius of the circle:

$$\theta[rad] = \frac{\ell_\theta}{R}$$

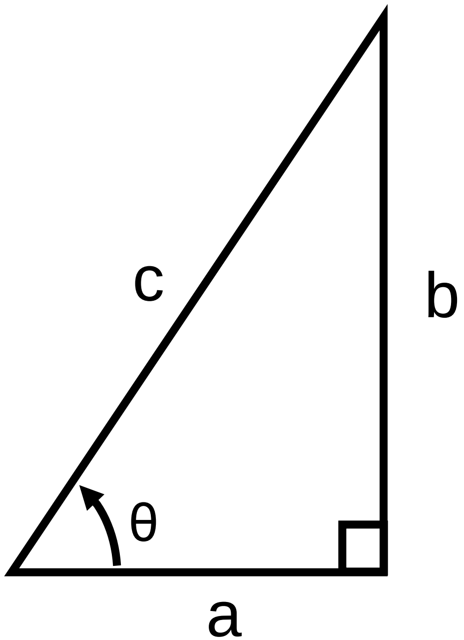
- For a full circle, $\ell_\theta = 2\pi R$, so $\theta = 2\pi$ Radians
- $360^\circ = 2\pi$ Radians

Right Triangles

- Pythagorean theorem: $$c^2 = a^2 + b^2$$

$$\sin\theta = \frac{b}{c} \quad , \quad \cos\theta = \frac{a}{c} \quad , \quad \tan\theta = \frac{b}{a}$$

$$\cos^2\theta + \sin^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$$



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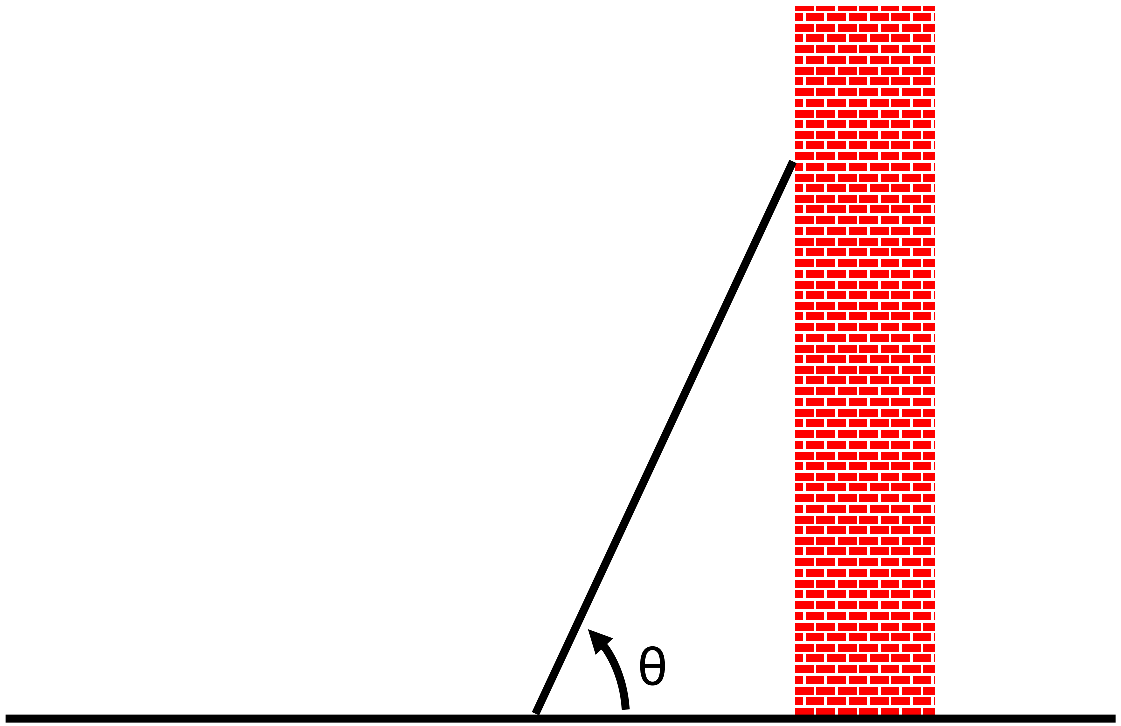
Example Problem

- A 4m ladder rests against a wall. You determine that the safest angle to keep the ladder at is 65º. How high does the ladder reach on the wall? How far from the wall is the base of the ladder?
- Steps to Solution:
  - Draw diagram
  - Label and put in all relevant information
  - Determine the math/physics which applies
  - Set up equation(s)
  - Solve equation(s)

Solving Example Problem

- Draw Diagram



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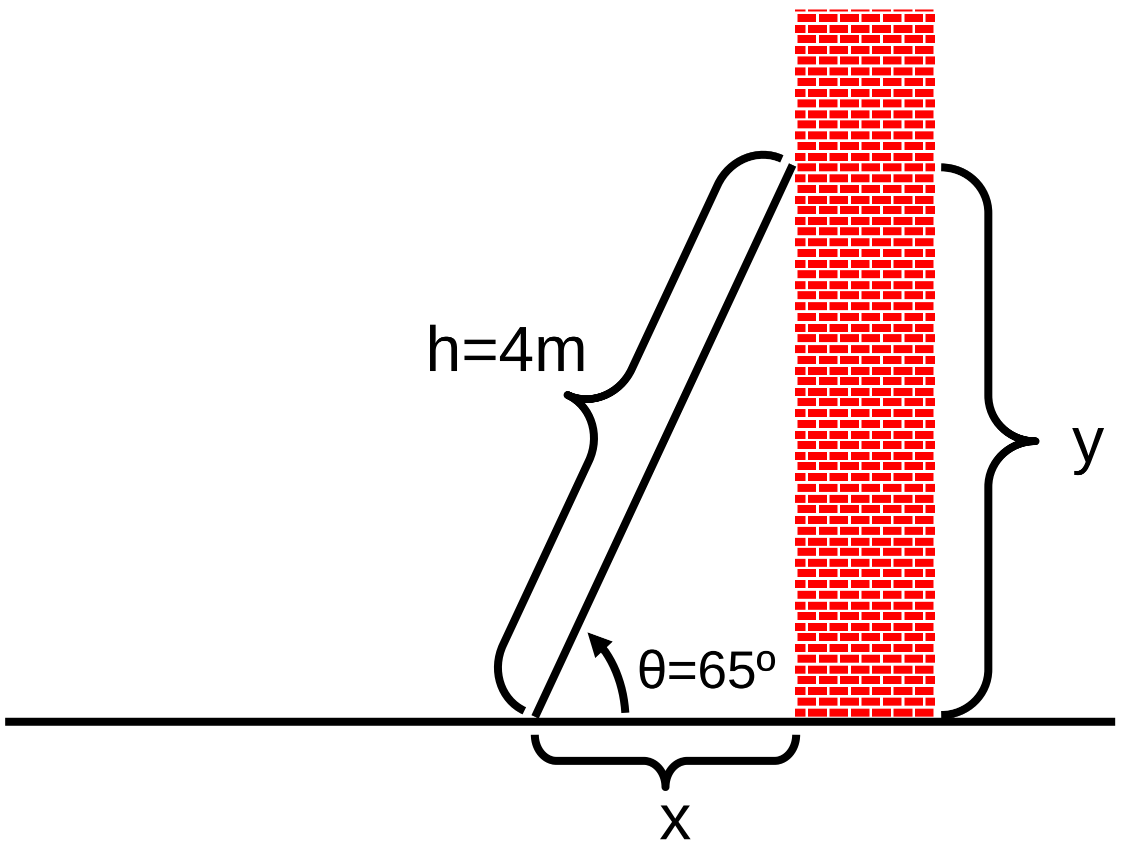
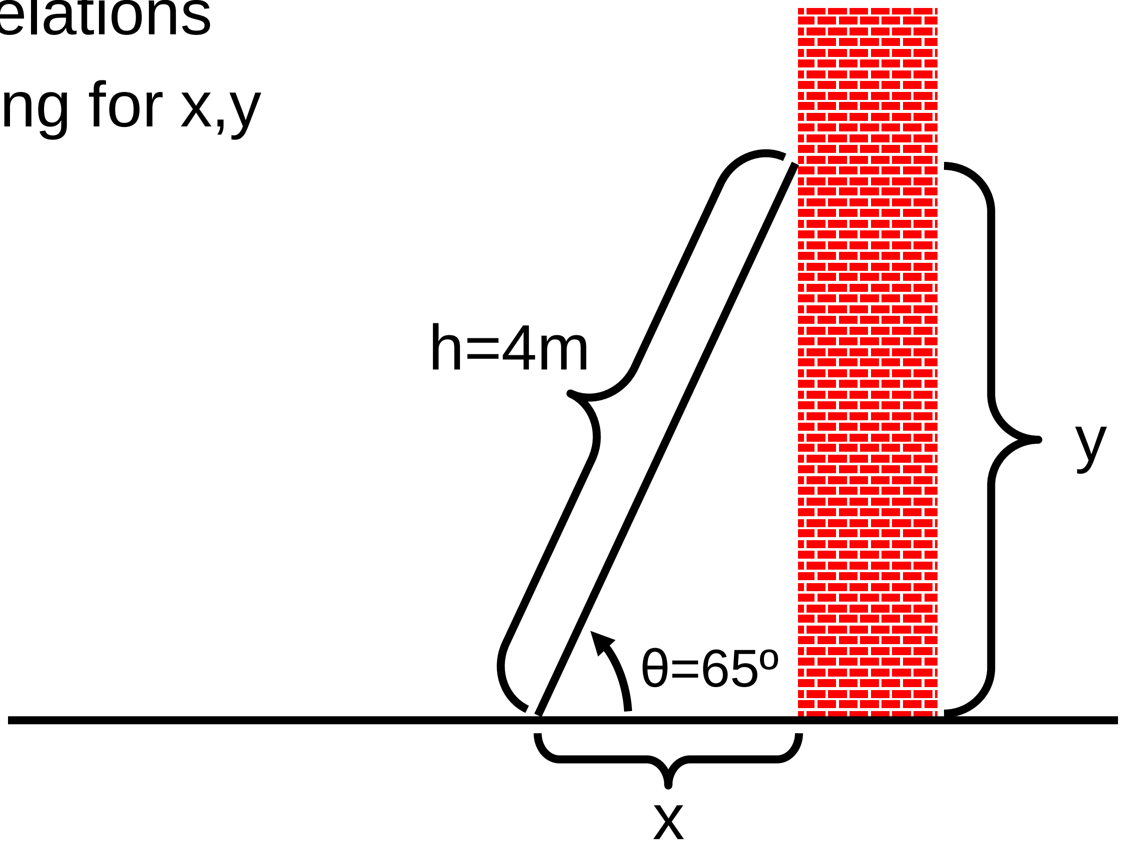
Solving Example Problem

- Label and put in all relevant information



Solving Example Problem

- Determine math/physics involved
  - Ladder/Wall make a right triangle → can use basic trig. relations
  - Looking for x,y



Solving Example Problem

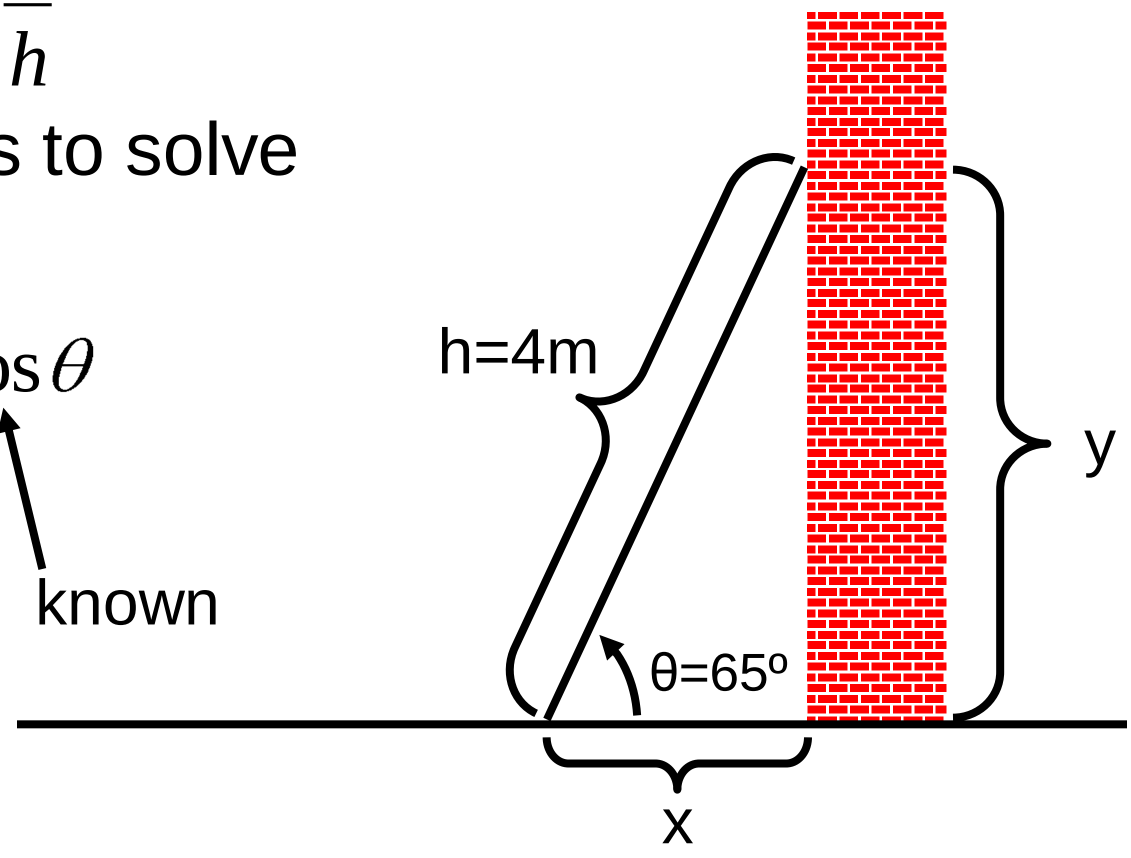
- Set up equations

$$\sin\theta = \frac{y}{h} \qquad \cos\theta = \frac{x}{h}$$

- Rewrite equations to solve for unknowns

$$y = h\sin\theta \qquad x = h\cos\theta$$

unknown known unknown known



Solving Example Problem

- Solve equations

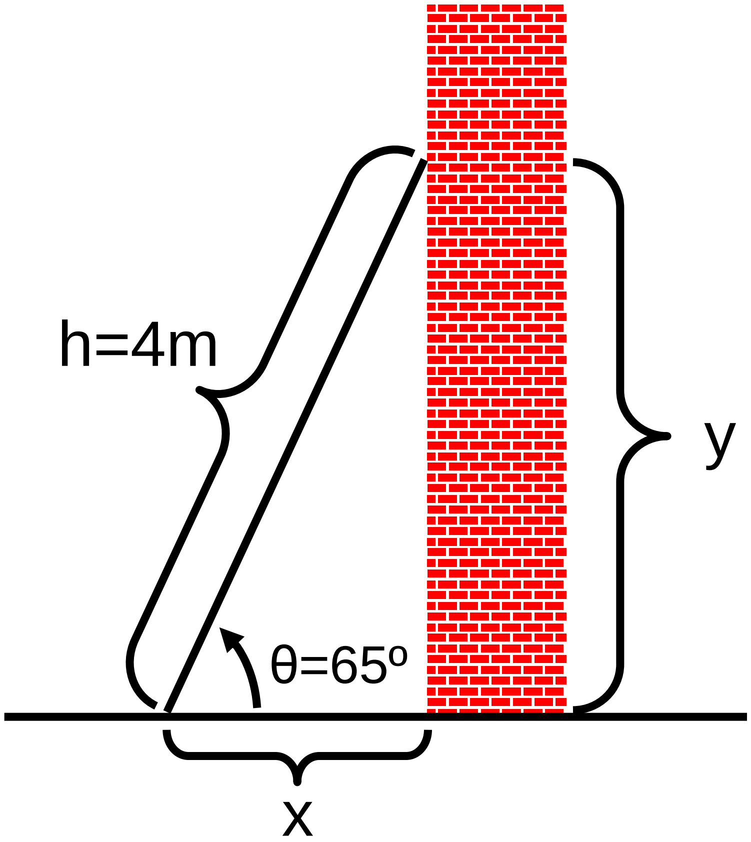
$$y = h\sin\theta \qquad x = h\cos\theta$$

- Input numbers

$$y = 4[m]\sin 65^\circ \quad x = 4[m]\cos 65^\circ$$

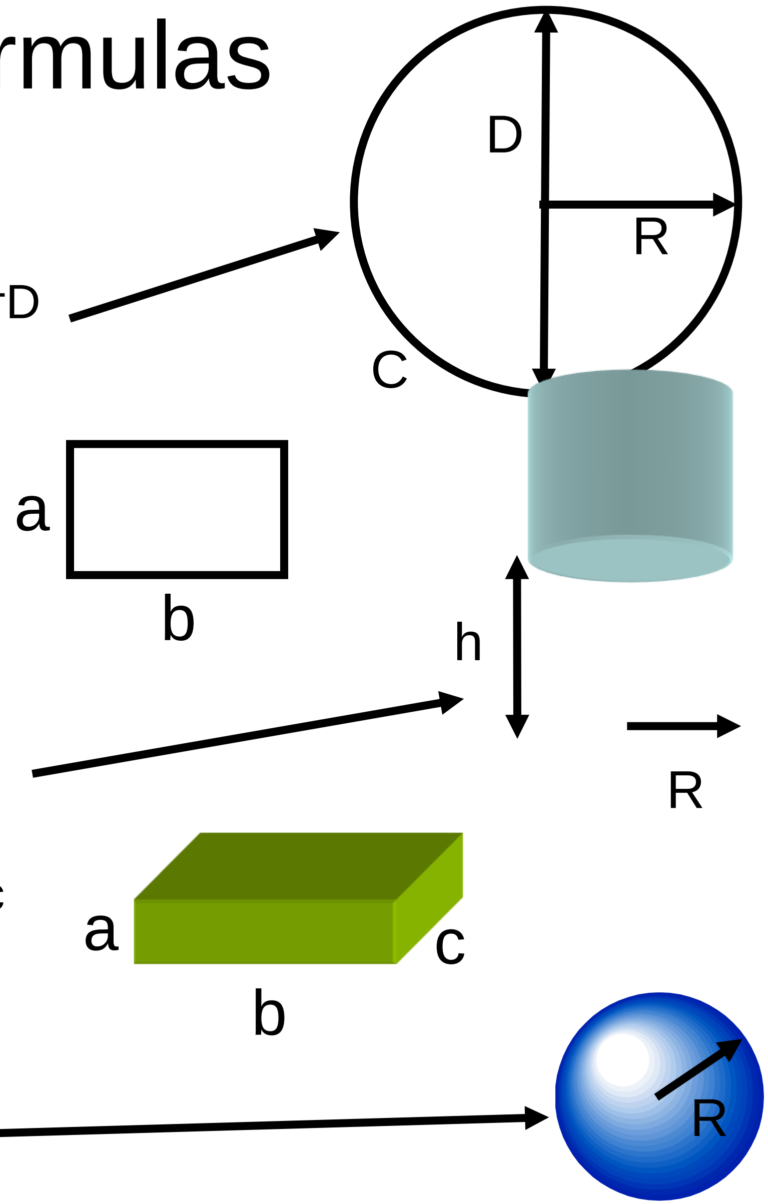
$$y = 3.63[m] \qquad x = 1.69[m]$$

- Be careful: does your calculator use degrees or radians? If radians, you must convert 65º to radians!



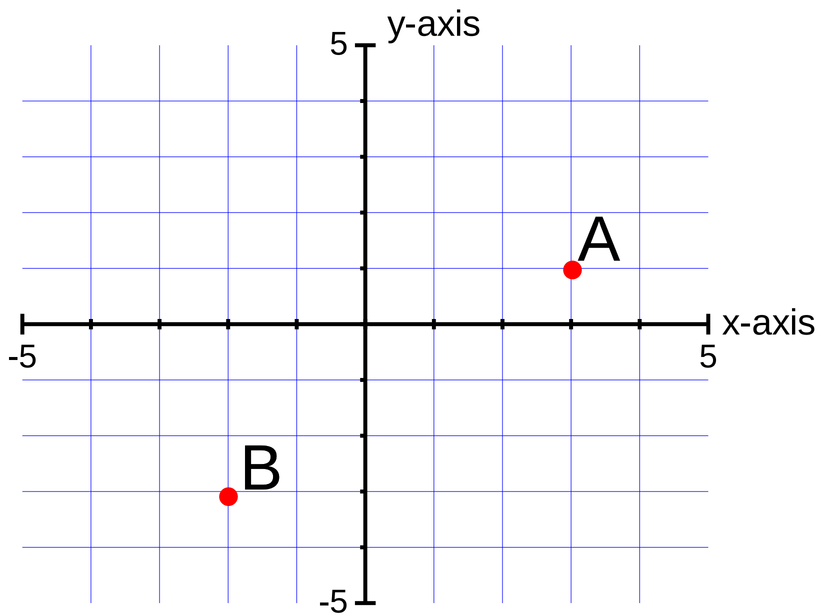
Useful geometric formulas

- Circle
  - C = circumference = $2\pi R = \pi D$
  - A = Area = $\pi R^2 = (\pi/4)D^2$
- Rectangle
  - P = perimeter = $2a + 2b$
  - A = Area = ab
- Right Cylinder
  - Surface Area = $2\pi Rh + 2\pi R^2$
  - Volume = $2\pi hR^2$
- Rectangular Box
  - Surface Area = $2ab+2ac+2bc$
  - Volume = abc
- Sphere
  - Surface Area = $4\pi R^2$
  - Volume = $(4/3)\pi R^3$



Coordinate Systems

- A coordinate system allows us to assign numbers to a point in space.
- The most common coordinate system is the Cartesian Coordinate system. Here, for two dimensions, we describe a point's position by its distance from two axes, the x and y axes.



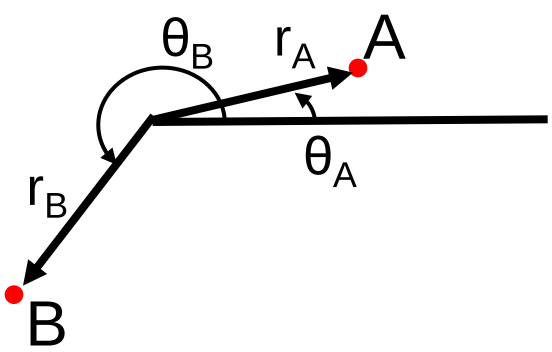
Point A → (3,1) 3 steps in the x-direction, 1 in the y, from (0,0), the origin.

Point B → (-2, -3) 2 steps in the negative x-direction, 3 in the negative y, from (0,0), the origin.

-We always write the x position first, followed by the y.

Polar Coordinates

- The polar coordinate system defines all points in terms of a distance (r) and an angle (θ).

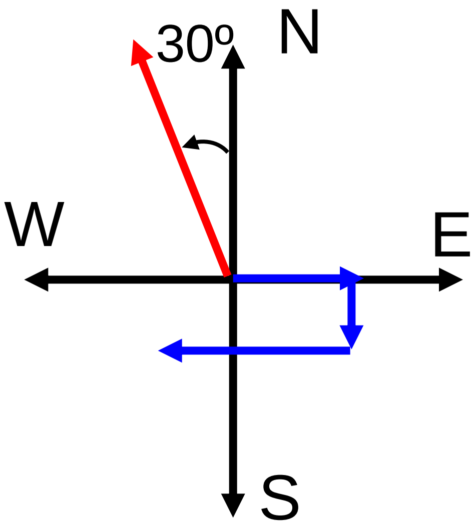


Point A → (3,1) in Cartesian coordinates, can be written as (3.16, 18.43º) in polar coordinates.

Point B → (-2, -3) in Cartesian coordinates can be written as (3.61, 236.3º).

Coordinate Systems

- One final coordinate system you are probably familiar with uses North, South, East, and West in place of +y, -y, +x, and –x.



For instance, to describe the direction a boat is traveling in, one might say "30º West of North" (red arrow)

Or…

You could describe a path you travel as:

3 miles E, 2 miles S, and 5 miles W. (blue arrows)

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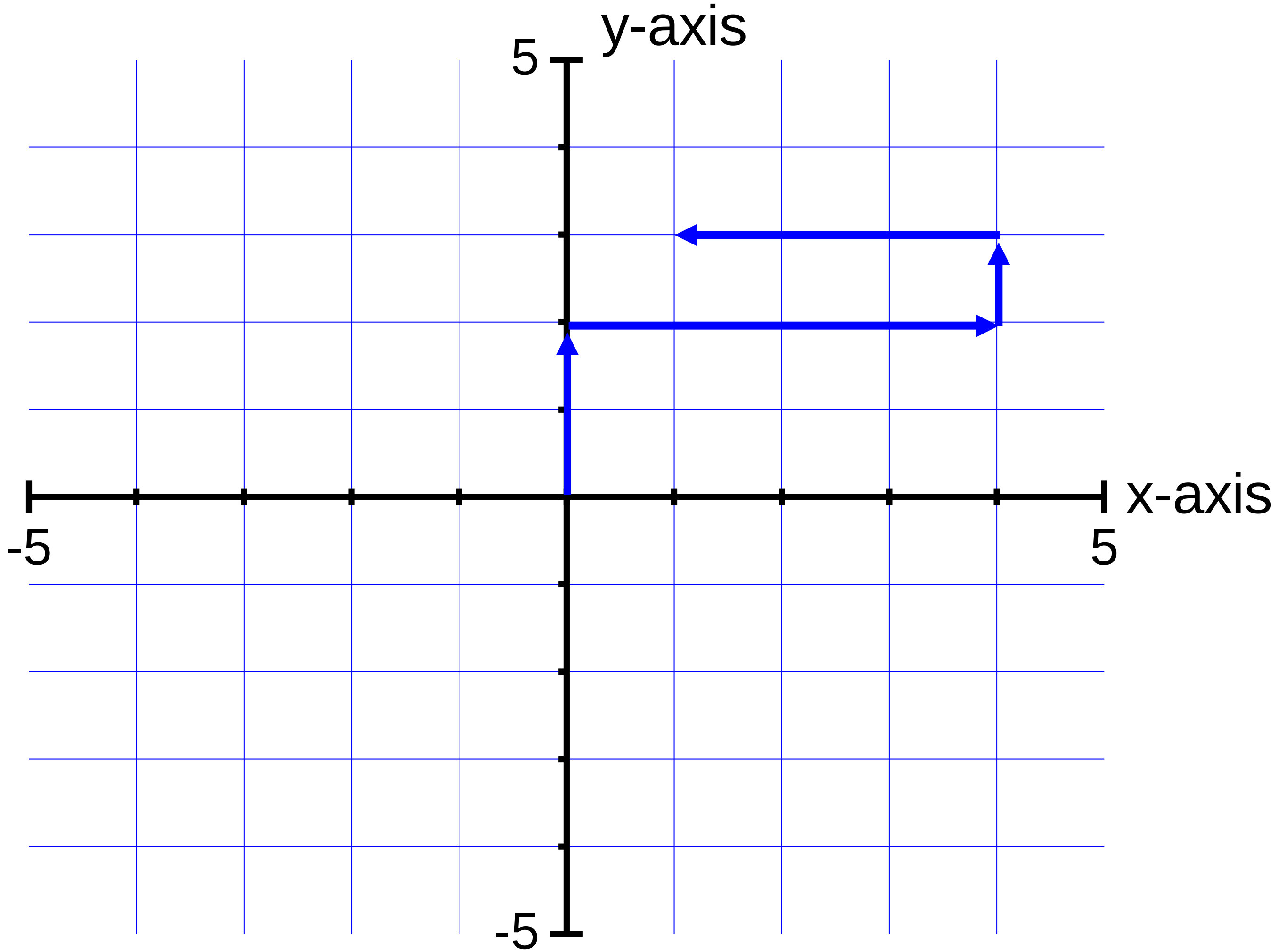
Practice Problem

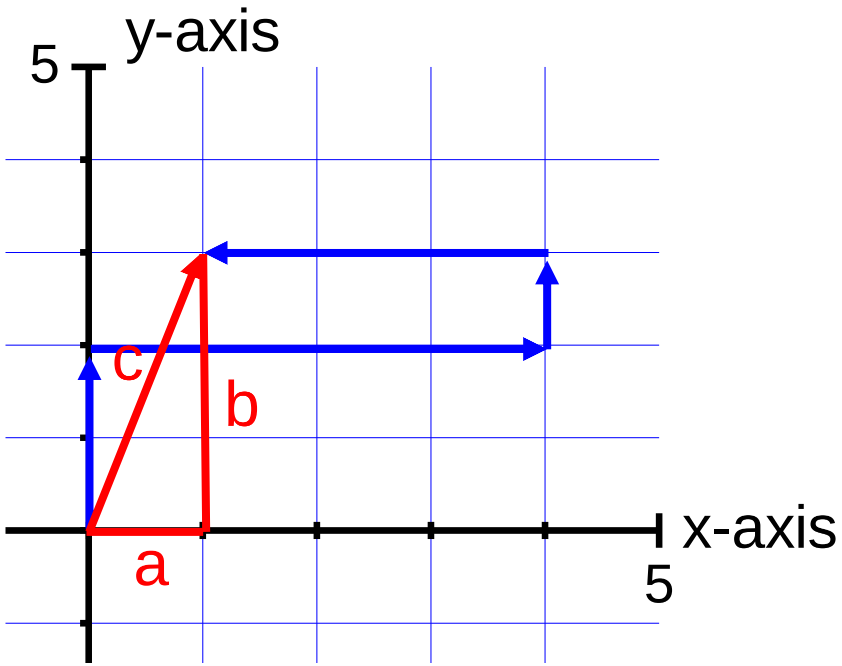
- A runner describes her running path as follows: 2 miles N, 4 miles E, then 1 mile N again, and finally 3 miles W.
- How far has the runner traveled, and how far is she from her starting point?
- Steps to Solution:
  - Draw diagram
  - Label and put in all relevant information
  - Determine the math/physics which applies
  - Set up equation(s)
  - Solve equation(s)

Draw Diagram



Solve Graphically

- By drawing the runner's path, you know she ends 3 miles N and 1 mile E of where she started.
- To find the total distance she ran, simply add up the lengths of each leg of her journey: 2 miles + 4 miles + 1 mile + 3 miles = 10 miles.

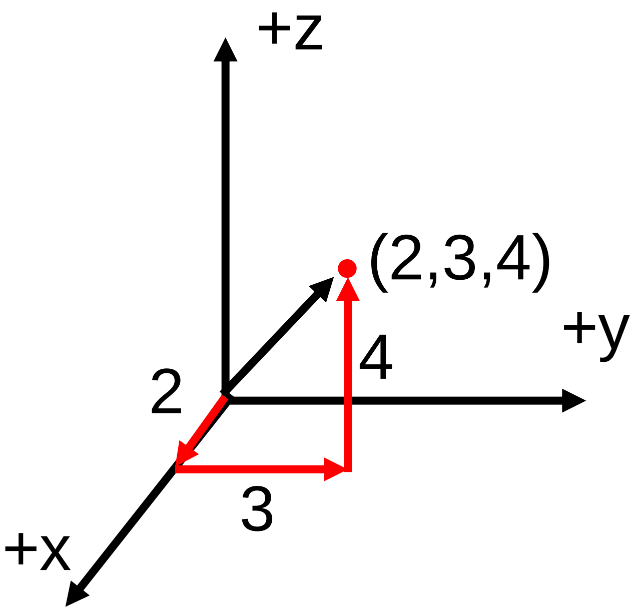


- Now draw an arrow from her starting position to her end position. The length of this arrow is the distance she ends from her starting point.
- If we know the coordinates of her final position, we can determine the length of the red arrow by making a right triangle and using

$$a^2 + b^2 = c^2 \quad or \quad c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 3^2} = \sqrt{5}$$

3D Coordinate systems

- Points in Three-Dimensional space are also easily described by the Cartesian coordinate system. But instead of just describing a point by its x and y coordinates, we add an axis, the z-axis, which describes the height of the point above the xy plane.



- The point (2,3,4) is shown to the left.
- For 3-dimensions, positions are written as (x,y,z)

Functions

- A function is a mathematical expression that gives you an output which is determined by one or more inputs.
- For instance: $f(x) = x^2$

Here f(x) is your output, it is determined by the value of the input (x).

If x=2, f(x)=4, if x=5, f(x)=25

- A function can have multiple inputs: $g(x, y) = \sin(x)e^{-y}$
- Or, a function can have both variables and constants:

$$f(t) = at^2 + bt + c$$

- Here a,b,c are constants, while t is a variable

Example: functions

- The position of a ball as a function of time is given by:

$$Position = (x(t), y(t)) \quad where$$

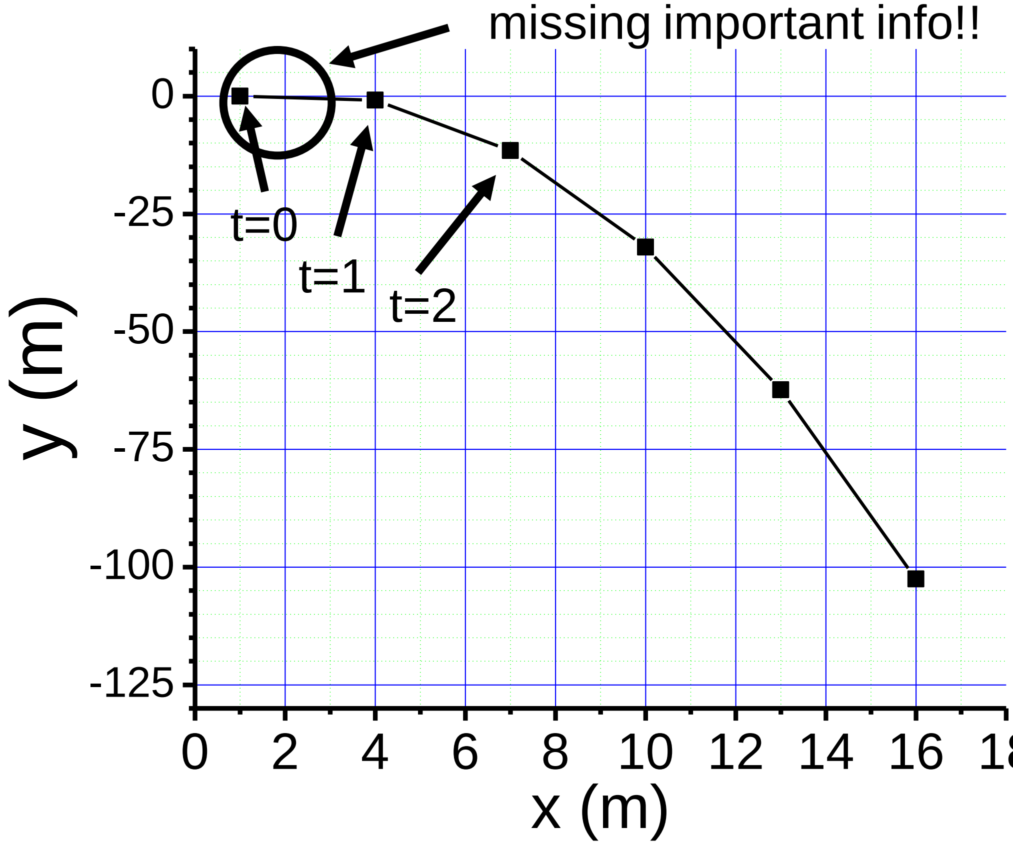
$$x(t) = 3[m/s] \times t + 1[m] \quad and \quad y(t) = -\frac{1}{2}(9.8[m/s^2]) \times t^2 + 4[m/s]$$

- Here, both the x and y position of the ball depend on time, or are function of time. The position of the ball (x,y)
- Plot the trajectory of the ball on a Cartesian coordinate system

Example: functions

- Create a table of x(t) and y(t) as a function of time

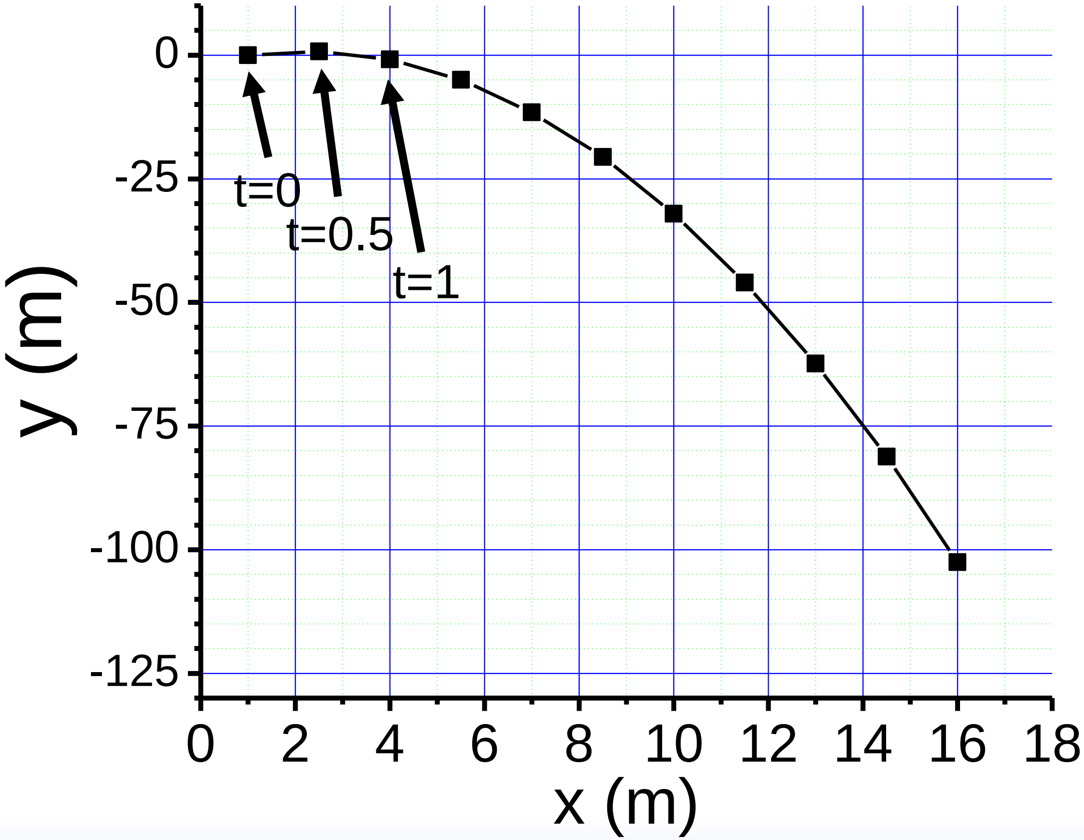
time (s)	x(t)	y(t)
0	1	0
1	4	-0.9
2	7	-11.6
3	10	-32.1
4	13	-62.4
5	16	-102.5



Example: functions

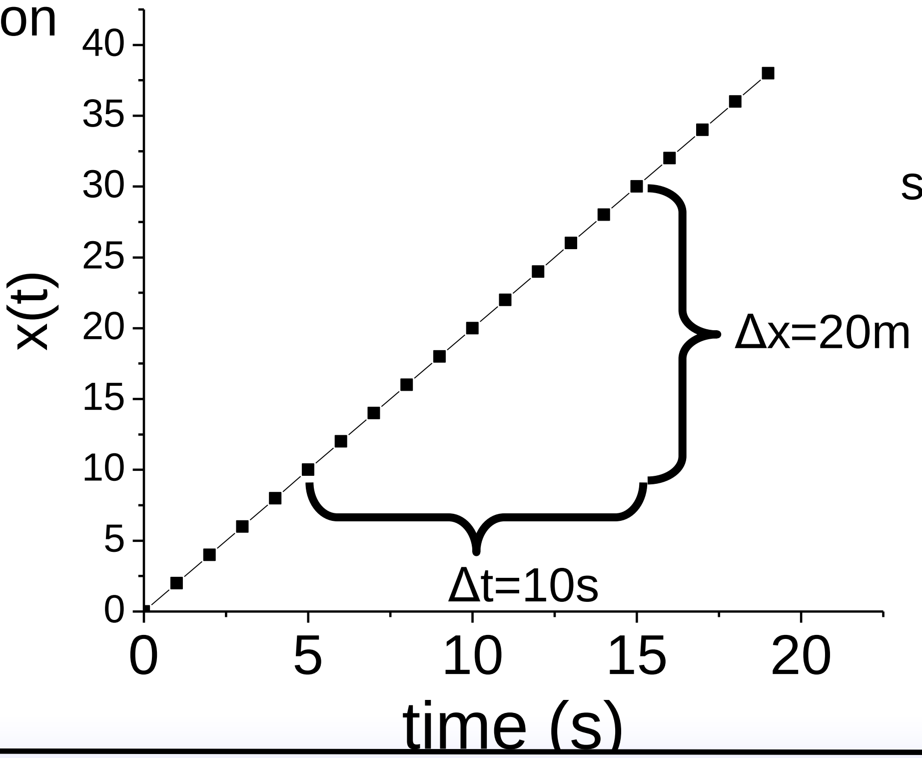
- I chose a bad time interval. If I go to a smaller time interval, 0.5 seconds, I see that the ball goes up before falling!!

time (s)	x(t)	y(t)
0	1	0
0.5	2.5	0.775
1	4	-0.9
1.5	5.5	-5.025
2	7	-11.6
2.5	8.5	-20.625
3	10	-32.1
3.5	11.5	-46.025
4	13	-62.4
4.5	14.5	-81.225
5	16	-102.5



Derivatives

- Oftentimes the slope of a function tells us very important information
- For instance, if you plot the position(x) of an object vs. time(t), the slope of this plot tells you the speed of the object
- Graphically, we can find the slope of a function by measuring the change in the y-axis position and dividing by the change in the x-axis position



speed =Δx/Δt=2m/s

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Derivatives

- Calculating the slope of a plot graphically is great if you have a simple function, one whose slope doesn't change much over time, but this is rare.
- Luckily, there is a mathematical method for calculating the slope of a function. It's called the derivative.
- We write the derivative of a function f(t) as f'(t) or df/dt
- Mathematically, the derivative is described as

$$f'(t) = df/dt = \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad as \quad \Delta t \to 0$$

- This looks complicated, but what is actually saying is that we take the slope Δf/Δt for infinitesimally small Δt. As Δt gets smaller and smaller, the derivative can tell us the slope of a line at a single point!!

Standard derivatives

- Below is a list of standard derivatives you will be expected to know for this class
- For a polynomial

$$f(x) = ax^n \quad , \quad f'(x) = anx^{n-1}$$

$$example: \quad f(x) = 3x^3 + x^2 + 4x + 2 + 2x^{-2}$$

$$f'(x) = 9x^2 + 2x + 4 + 0 - 4x^{-3}$$

- For sin or cos

$$f(x) = A\cos(bx) \quad , \quad f'(x) = -Ab\sin(bx)$$

$$f(x) = A\sin(bx) \quad , \quad f'(x) = Ab\cos(bx)$$

$$example: \quad f(x) = 3\cos(x) + \sin(2x)$$

$$f'(x) = -3\sin(x) + 2\cos(2x)$$

- For exponential

$$f(x) = Ae^{bx} \quad , \quad f'(x) = Abe^{bx}$$

$$example: \quad f(x) = 4e^{3x} \; , \quad f'(x) = 12e^{3x}$$

Rules for Derivatives

- Derivative of a sum of functions

$$z(x) = f(x) + g(x)$$

$$\frac{dz}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

$$z(x) = 3x^2 + e^{3x}$$

$$\frac{dz}{dx} = 6x + 3e^{3x}$$

- Derivative of a product of functions (product rule)

$$z(x) = f(x)g(x)$$

$$\frac{dz}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$

$$z(x) = 4\sin(x)\cos(x)$$

$$\frac{dz}{dx} = 4\cos(x)\cos(x) - 4\sin(x)\sin(x)$$

- Chain rule

$$z(x) = f(g(x))$$

$$\frac{dz}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$z(t) = 3e^{2x} + \cos(x) \quad , \quad x = t^2$$

$$\frac{dz}{dt} = \frac{df}{dx}\frac{dx}{dt} = (6e^{2x} - \sin(x))(2t) = 12te^{2t^2} - 2t\sin(t^2)$$

The Second Derivative

- Taking the second derivative of a function can also give us valuable information. For instance the second derivative of position with respect to time gives acceleration. The second derivative is found by taking the derivative of the derivative of a function. We write the second derivative of a function f(t) as f''(t) or d^2f/dt^2.

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$if \quad f(x) = 4x^3 + 3x^{-1}$$

$$f'(x) = 12x^2 - 3x^{-2}$$

$$f''(x) = 24x + 6x^{-3}$$

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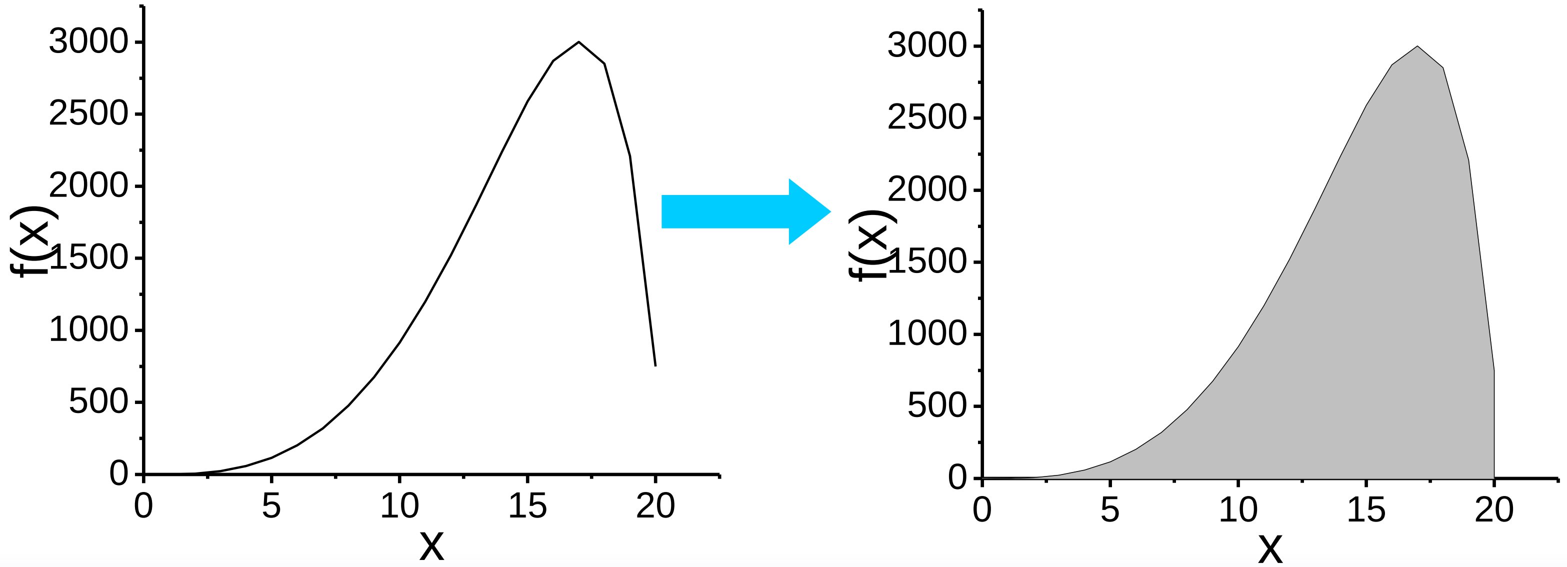
Integrals

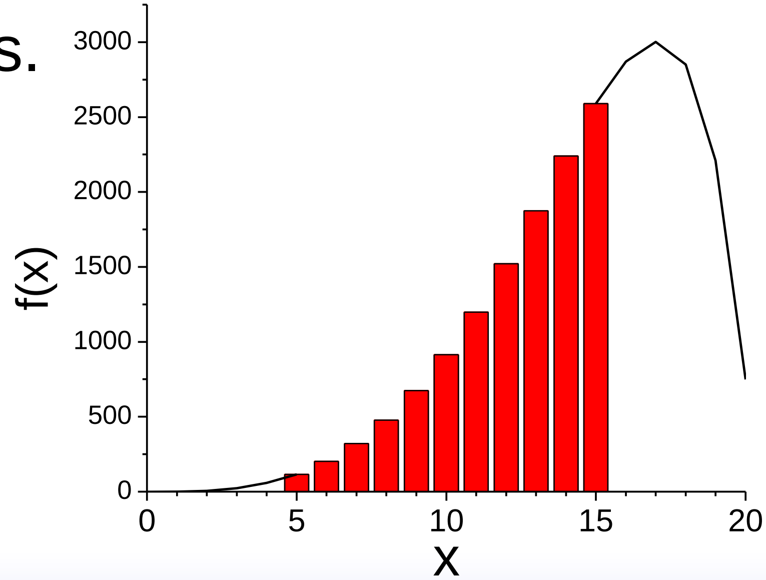
- Integrals are sometimes referred to as *antiderivatives,* and we will see why in the next slide.
- Integrals also have a graphical meaning:
- If you take the integral of a function, you are finding the *AREA* underneath that function's curve;



Integrals

- The process of calculating the area under a plot, between point x_1 and x_2, can be understood in the following way:
  - Divide your graph into many rectangles of width Δx. Each rectangle will have an area $dA=f(x)\Delta x$.
  - Add up all of these rectangles.
  - The plot here shows the sum of rectangles from x=5-15.

$$\sum f(x)\Delta x$$

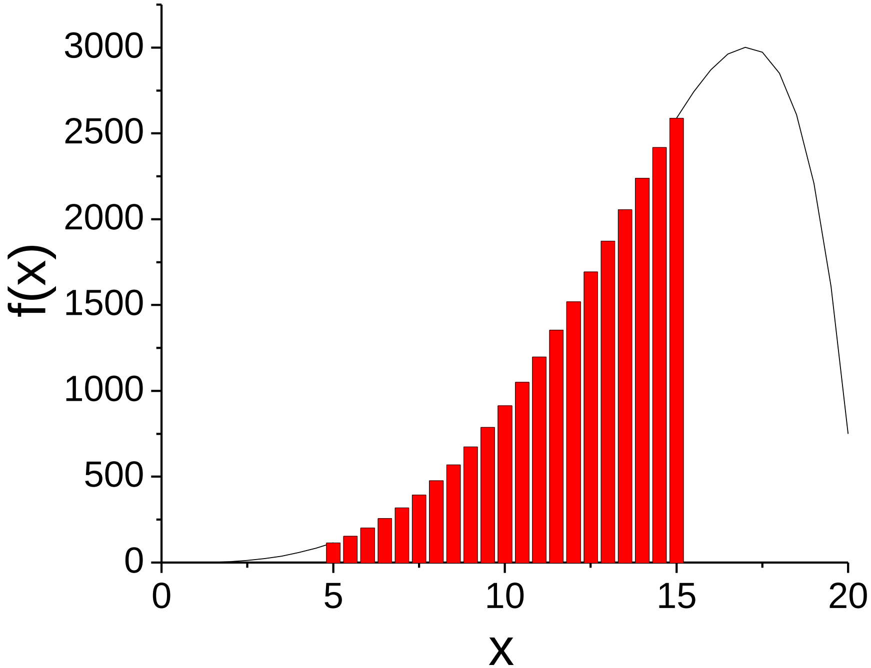


Integrals

- Dividing into rectangles and summing is an approximation of the area under the curve.
- To improve the approximation, you would make smaller and smaller rectangles.

- The plot here is the same as the previous page, except Δx is half the size, making this sum twice as accurate.

$$\sum f(x)\Delta x$$



Integrals

- Eventually, to get perfect accuracy, you take $\Delta x \rightarrow 0$. Then this happens, you no longer have a sum, but an integral. We write the expression for an integral as:

$$\int_{x_1}^{x_2} f(x)dx$$

- Integrals are sometimes called antiderivatives. This is because if you take the integral of a derivative of a function, you get back the original function.

$$f(x) = \int f'(x)dx$$

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Standard Integrals

- Below is a list of standard integrals you will be expected to know for this class
- For a polynomial

$$f(x) = ax^n \quad , \quad \int f(x)dx = \frac{ax^{n+1}}{n+1}$$

$$example: \quad f(x) = 3x^3 + x^2 + 4x + 2 + 2x^{-2}$$

$$\int f(x)dx = \frac{3}{4}x^4 + \frac{1}{3}x^3 + 2x^2 + 2x - 2x^{-1}$$

- For sin or cos

$$f(x) = A\cos(bx) \quad , \quad \int f(x)dx = \frac{A}{b}\sin(bx)$$

$$f(x) = A\sin(bx) \quad , \quad \int f(x)dx = -\frac{A}{b}\cos(bx)$$

$$example: \quad f(x) = 3\cos(x) + \sin(2x)$$

$$\int f(x)dx = 3\sin(x) - \frac{1}{2}\cos(2x)$$

- For exponential

$$f(x) = Ae^{bx} \quad , \quad \int f(x)dx = \frac{A}{b}e^{bx}$$

$$example: \quad f(x) = 4e^{3x} \, , \quad \int f(x)dx = \frac{4}{3}e^{3x}$$

Differential Equations

- You are probably familiar with solving standard equations, for instance the quadratic equation:

$$ax^2 + bx + c = 0 \quad , \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- But sometimes you will see equations of functions, for instance:

$$\frac{d^2 f(x)}{dx^2} = b^2 f(x)$$

$$\frac{d^2 f(x)}{dx^2} = -a^2 f(x)$$

- This solutions to these equations are functions which when you take their second derivative, returns the original equation multiplied by a constant

Differential Equations

- These differential equations are actually quite simple, if you think about what we have already discussed about derivatives.
- For the first equation: $$\frac{d^2 f(x)}{dx^2} = b^2 f(x)$$
- We know: $$\frac{d(Ae^{bx})}{dx} = bAe^{bx} = bf(x)$$
- And if we take a second derivative:

$$\frac{d^2(Ae^{bx})}{dx^2} = \frac{d(bAe^{bx})}{dx} = b^2 Ae^{bx} = b^2 f(x)$$

- So the solution to this differential equation is: $f(x) = Ae^{bx}$

Differential Equations

- For the second equation:

$$\frac{d^2 f(x)}{dx^2} = -a^2 f(x)$$

- We know: $$\frac{d(A\sin(ax))}{dx} = aA\cos(ax)$$

- And if we take a second derivative:

$$\frac{d^2(A\cos(ax))}{dx^2} = \frac{d(aA\cos(ax))}{dx} = -a^2 A\sin(ax) = -a^2 f(x)$$

- So the solution to this differential equation is: $f(x) = A\sin(ax)$

- Show that $f(x) = A\cos(ax)$ is also a solution!

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