22.520 Numerical Methods for PDEs: Video 21: The Weak Form

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The following materials were used in the preparation of this lecture:

1. 16.920, Lecture Notes
2. Strang & Fix: An analysis of the Finite Element Method
3. Bathe, Hughes, etc. have introductory books that are useful.
4. Additional link to MIT 19.901, Undergraduate Numerical Methods Course – may be useful.

http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-901-computational-methods-in-aerospace-engineering-spring-2005/ The author of these slides wishes to thank these sources for making the current lecture.
Weak Form

- In finite elements we consider the *weak form* of the governing equation.
- Consider a general PDE, represented by the functional $\mathcal{L}(u)$:
  \[
  \mathcal{L}(u) = f
  \tag{1}
  \]
- With some boundary conditions $u_{\Gamma_D} = g$ and $\frac{\partial u}{\partial n}|_{\Gamma_N} = h$
- What if we multiply this by a function $w$ and integrate across the domain of interest:
  \[
  \int_{x_L}^{x_R} w \mathcal{L}(u) dx = \int_{x_L}^{x_R} w f dx
  \tag{2}
  \]
- Solving this expression is effectively the same as solving the original equation.
- **BUT** what does it mean to do this?
Weak Form

- Graphically, it shows that we are satisfying a weighted expression of the equation in the domain/sub-domain:

\[ \mathcal{L}(u) = f \]

\[ \mathcal{L}(u) - f = R \]

\[ \int_{W} \mathcal{L}(u) \, dx - \int_{W} f \, dx = \int_{R} R \, dx = 0 \]

- Since we are satisfying a weighted form of the equation, we can accentuate the expression as desired.
Let's do an example with the Poisson equation/Laplace operator in one dimension:

\[
\frac{\partial^2 u}{\partial x^2} = f
\]  

(3)

Let's re-arrange the equation a little:

\[
\frac{\partial^2 u}{\partial x^2} - f = R = 0
\]  

(4)

This shows that the equation has a zero residual, \( R \) when solved exactly.
Let’s now multiply the expression by $w$, a weighting function and integrate across the domain/region of interest:

$$
\int_{x_L}^{x_R} \left( w \frac{\partial^2 u}{\partial x^2} \right) dx - \int_{x_L}^{x_R} (wf) dx = \int_{x_L}^{x_R} wR dx = 0 \quad (5)
$$

The result is an integral expression over the domain.

Notice, that the residual has been weighted and integrated – and remains zero.
The weak form

- Let’s integrate the weighted residual integral expression by parts:
  \[
  \int_{x_L}^{x_R} \left( w \frac{\partial^2 u}{\partial x^2} \right) \, dx - \int_{x_L}^{x_R} (wf) \, dx = 0
  \] (6)

- Recall, integration by parts says:
  \[
  \int_{x_L}^{x_R} f(x)g'(x) \, dx = f(x)g(x) \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} f'(x)g(x) \, dx
  \] (7)

- Let’s set \( f(x) = w \) and \( g'(x) = \frac{\partial^2 u}{\partial x^2} \), then:
  \[
  \int_{x_L}^{x_R} \left( w \frac{\partial^2 u}{\partial x^2} \right) \, dx = w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx
  \] (8)
The weak form

- Re-inserting this into the equation 6:

\[
\int_{x_L}^{x_R} \left( w \frac{\partial^2 u}{\partial x^2} \right) dx - \int_{x_L}^{x_R} (wf) \, dx =
\]

\[
w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx - \int_{x_L}^{x_R} (wf) \, dx = 0
\]

- Let's focus only on the reformulated expression:

\[
w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx - \int_{x_L}^{x_R} (wf) \, dx = 0 \quad (9)
\]

- Which we will call the weak form.
The weak form

- The final weak form of the equation is:

\[ w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx - \int_{x_L}^{x_R} w f \, dx = 0 \]  

(10)

- \( a(w, u) \): Is the bilinear form
- Often this is written as:

\[ a(u, w) - f(w) = 0 \]  

(11)

- This final weak form is integrals of first derivatives, whereas the strong form was second derivatives.
What have we learned

Let’s look at the graphic again:

\[ \mathcal{L}(u) = f \]

\[ \mathcal{L}(u) - f = R \]

\[ \int W \mathcal{L}(u) dx - \int W f dx = \int W R dx = 0 \]

Integration by parts results in a reduction of order of PDE.