22.520 Numerical Methods for PDEs: Video 22: One Dimensional FEM Nodal Discretization

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The following materials were used in the preparation of this lecture:

1. 16.920, Lecture Notes
2. Strang & Fix: An analysis of the Finite Element Method
3. Bathe, Hughes, etc. have introductory books that are useful.
4. Additional link to MIT 19.901, Undergraduate Numerical Methods Course – may be useful.

http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-901-computational-methods-in-aerospace-engineering-spring-2005/  The author of these slides wishes to thank these sources for making the current lecture.
Planning a discrete FEM

Let’s look at the steps we took for finite differences – Poisson’s equation:

1. Discretize the domain
2. Discretize the equation
3. Write an equation for each node of the domain
4. Construct an 'A-matrix'
5. Construct a RHS matrix
6. Set boundary conditions
7. Solve, post-process

We’ll take similar steps, while converting theoretical development into an implementation.
We begin as usual, and discretize the domain into $N$-intervals and $N + 1$ nodes:

Note, we don’t need to have equally spaced intervals.
Discretize the domain

- When we divide the domain into intervals, we will need to make sure that the weak form of the equation is still satisfied on each element/interval:

\[
\left. w \frac{\partial u}{\partial x} \right|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx - \int_{x_L}^{x_R} w f \, dx = 0 \tag{1}
\]

- We can write the integral for each element, \( elem \):

\[
\left( w \left. \frac{\partial u}{\partial x} \right|_{x_L}^{x_R} \right)_{elem} - \left( \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \, dx \right)_{elem} - \left( \int_{x_L}^{x_R} w f \, dx \right)_{elem} = 0 \tag{2}
\]
Discretize the domain

- The boundary of integration terms:
  \[ w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} \]
  (3)

- Cancel out from one element to the next because
  \[ w \frac{\partial u_i}{\partial x} \bigg|_{x_L} = w \frac{\partial u_{i-1}}{\partial x} \bigg|_{x_R}. \]

- But, these limits of integration terms remain on the boundaries of the domain where there are no elements for cancellation.

\[ w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \left( \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \right)_{\text{elem}} dx - \int_{x_L}^{x_R} (wf)_{\text{elem}} dx = 0 \]
(4)
Linear Hat Basis functions for the $w$ and $u$

- We will introduce linear basis functions to represent the solution $u$ and the test function $w$.
- There will be a test function $w_j$ associated with each node $j$.
- There will be a series of basis functions $\phi_i$ for constructing the solution across the domain.
Recall: 1-D Linear Hat Basis function

- The linear hat basis function will span two elements in the domain.
- The value of the basis function associated with an arbitrary node $i$ is:

\[
\phi_i^+(x) = \frac{1}{h_n}(\xi - x_i) \quad x_i < \xi < x_{i+1} 
\]

\[
\phi_i^-(x) = \frac{1}{h_{n-1}}(\xi - x_{i-1}) \quad x_{i-1} < \xi < x_i 
\]

\[
\phi_i = 0 \quad \text{elsewhere}
\]
At times, it is simpler to analyze each element $i$ independently:

$$\phi_{i+1}(\xi) = \frac{1}{h}(\xi) \quad 0 < \xi < h$$  \hspace{1cm} (8)

$$\phi_i(\xi) = \frac{1}{h}(h - \xi) \quad 0 < \xi < h$$  \hspace{1cm} (9)

... and then save the information in terms of nodes.
Basis Functions for Representing the Solution

- Let’s introduce \( \hat{u}(x) = \sum_{j=1}^{NV} u_i \phi_j(x) \)
- Also, let’s introduce \( w(x) \) for each node, \( w_i(x) = \phi_i(x) \)

\[
\begin{align*}
  w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \left( \sum_{j=1}^{NV} u_j \phi_j(x) \right)}{\partial x} \right) dx - \int_{x_L}^{x_R} (\phi_i f) \, dx &= 0 \quad (11) \\
  &
\end{align*}
\]

- Let’s rearrange things a little. The things that are known \((w, f)\) go to the right hand side, while the unknowns remain on the LHS:

\[
\begin{align*}
  w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \int_{x_L}^{x_R} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \left( \sum_{j=1}^{NV} u_j \phi_j(x) \right)}{\partial x} \right) dx = \int_{x_L}^{x_R} (\phi_i f) \, dx \quad (12)
\end{align*}
\]

- Remember: this is the equation we must satisfy for each node! We have one equation per test function (here represented as \( i \)).
Let’s rearrange things even more (pull the $u_j$ and $\sum$ out of integration):

$$w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \sum_{j=1}^{NV} u_j \int_{x_L}^{x_R} \left( \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x} \right) dx = \int_{x_L}^{x_R} (\phi_i f) dx$$

Which is still the equation for the $i-th$ node. But what is:

$$\frac{\partial (\phi_i)}{\partial x} \quad \text{or} \quad \frac{\partial (\phi_j)}{\partial x} \quad \text{or} \quad \frac{\partial (\phi_j)}{\partial x} \cdot \frac{\partial (\phi_i)}{\partial x}$$

(13)
The derivatives of basis functions are \( \frac{1}{h} \) and \( -\frac{1}{h} \) in different elements respectively:

So, how do we calculate the expression:

\[
\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x}
\]

(14)
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\[
\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x}
\]

(15)

\[\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x} = 0\]

\[\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x} = 0\]

\[\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x} = (1/h) \cdot (-1/h) = -1/(h^*h)\]
So, when $j = i - 1$ and $j = i + 1$ then \[
\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x} = -\frac{1}{h^2}
\]
And, when $j = i$ then \[
\frac{\partial (\phi_i)}{\partial x} \cdot \frac{\partial (\phi_j)}{\partial x} = \frac{2}{h^2}
\]
Does this look at all familiar?
It looks like \[
\frac{-u_{j-1} + 2u_j - u_{j-1}}{\Delta x^2}.
\]
Lot’s of math for the same result….let’s persist a few more steps.
A-Matrix Theory

- So, our original discrete expression is:

\[
 w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} - \sum_{j=1}^{NV} u_j \int_{x_L}^{x_R} \left( \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x} \right) \, dx = \int_{x_L}^{x_R} (\phi_i f) \, dx
\]

- When \( i = j \) for an internal node, and \( h \) is the same throughout the domain:

\[
 -u_j \int \left( \frac{2}{h^2} \right) \, dx = -u_j \left( \frac{2}{h^2} \right) \cdot h
\]

- When \( i = j - 1 \) or \( i = j + 1 \) for an internal node:

\[
 -u_j \int \left( \frac{-1}{h^2} \right) \, dx = -u_j \left( \frac{-1}{h^2} \right) \cdot h
\]
So for a given test function $i$, the internal node, weak form expression is:

$$-u_{j-1} \left( \frac{-1}{h^2} \right) \cdot h - u_j \left( \frac{2}{h^2} \right) \cdot h - u_{j+1} \left( \frac{-1}{h^2} \right) \cdot h = \int_{x_L}^{x_R} (\phi_i f) \, dx \quad (16)$$

So, this is the equation we use for each internal node of the domain/each row of the A-matrix. With the RHS integration now (for RHS vector):

$$-u_{j-1} \left( \frac{-1}{h^2} \right) \cdot h - u_j \left( \frac{2}{h^2} \right) \cdot h - u_{j+1} \left( \frac{-1}{h^2} \right) \cdot h = f \cdot h \quad (17)$$
Boundary conditions

Let’s now look at boundary conditions:

\[ w \frac{\partial u}{\partial x} \bigg|_{x_L}^{x_R} \]

(18)

- If \( \frac{\partial u}{\partial x} = 0 \) on the boundary, we can assume the above term equals zero.
- If \( u = 0 \) on the boundary, we must alter the A-matrix to include this value.
Solve for the Unknown

- At this point we solve for the unknown, $u$ and plot the result.
What have we learned

- We’ve looked at how we can implement a Finite Element Method (mathematically).
- Starting form the weak form:
  1. Discretize the domain into elements and nodes
  2. Break the weak form equation into an expression for each weighting function. Each weighting function expression becomes an equation in the system of unknowns.
  3. Determine how the weighting function interacts with the solution basis function to find the integral expression.
  4. Construct the A-matrix
  5. Construct the RHS
  6. Solve

- In the next module, we’ll look at implementation (a nodal and elemental implementation).