Panel Methods : Mini-Lecture

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3D - Panel Method Examples
Other Applications of Panel Methods

http://www.flowsol.co.uk/

http://oe.mit.edu/flowlab/
Boundary Element Methods

• Panel methods belong to a broader class of methods called Boundary Element Methods (BEM).

• Used in
  – Capacitance extraction
  – Structural analysis (some)
  – Drug design
  – MEMs Stokes flow
Start with the Basic Idea

- An initial thought experiment with vortices.
Let’s Start by Looking At This Problem

- Flow tangency: The equation we want to satisfy at each control point $j$, on the airfoil:

$$\vec{V}_\infty \cdot \hat{n}_j + \sum_{i=1}^{NP} (\vec{v}_{\Gamma_i} \cdot \hat{n}_j) = 0$$
Example of the Flow Around An Airfoil With **10** Point Vortices

Flow tangency **is** satisfied at the control point...
Example of the Flow Around An Airfoil With 50 Point Vortices
Example of the Flow Around An Airfoil With 100 Point Vortices
In search of a more elegant solution

Rather than discrete vortices, use a collection of vortex sheets (aka. panels) on the foil.

\[ \bar{v}(p) = -\sum_{i=1}^{NV} \left( \frac{\Gamma_i}{2\pi R_{i,p}} \right) \hat{e}_\theta \]

\[ v_\gamma(p) = \int_{x'} \nabla \left( \frac{\gamma(x')\theta(p, x')}{2\pi} \right) dx' \]
Point Vortex vs. Vorticity Distributions

Point Vortex

Constant Vorticity Distr.

Linear Vorticity Distr.
Linear Vortex Panel Method
Geometry → Panels

- **Step 1:** Get the geometry (airfoil, sphere, etc).
- **Step 2:** Discretize the geometry into panels (or computationally discrete elements)

Reality

![Diagram](image)

- NACA 4412
- Panel normal
- Panel Centroid = Evaluation Pt.
- Panel/Element
- Vertex/Node

Computational Representation of the geometry
Small Note On Mesh Generation

- Many numerical methods distribute discrete elements near regions of large changes (either in solution value or geometry). Panel Methods are no different. We usually want to cluster panels near regions where the flow changes rapidly or geometry has significant curvature.

<table>
<thead>
<tr>
<th>40 Equal Sized Panels</th>
<th>L.E. Clustering</th>
<th>L.E. and T.E. Clustering</th>
</tr>
</thead>
</table>

![Graphs showing mesh clustering patterns](image)
Step 3: Place a linearly varying distribution of vorticity on each panel

Using a panel method, we are trying to determine the vorticity distribution around the airfoil. By determining $\gamma_1 \rightarrow \gamma_{N+1}$ we have solved the problem. The value of the Vorticity sheet strength varies linearly between the values on each vertex of the panel.
**Step 4 : Building the Influence Matrix**

\[
\begin{pmatrix}
\vdots \\
V_{n}^{N,1} \\
V_{n}^{5,1} \\
V_{n}^{4,1} \\
V_{n}^{3,1} \\
V_{n}^{2,1} \\
V_{n}^{1,1} \\
0
\end{pmatrix}
+ \begin{pmatrix}
V_{n}^{1,1} \\
V_{n}^{2,1} \\
V_{n}^{3,1} \\
V_{n}^{4,1} \\
V_{n}^{5,1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
V_{n}^{N,1}
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\vdots \\
\vdots \\
\vdots \\
\gamma_N \\
\gamma_{N+1}
\end{pmatrix}
\]

\[
\vec{v}_{n}^{i,j} = \int_{x'} \hat{n}_{j} \cdot \nabla \left( \frac{\gamma(x') \theta(p, x')}{2\pi} \right) dx' = 0
\]
Step 4: Building the Influence Matrix

\[
\begin{pmatrix}
V_{n_{\text{inf}}} \\
\vdots \\
V_{n_{N,1}} \\
\vdots \\
V_{n_{N,2}}
\end{pmatrix}
+ \begin{pmatrix}
V_{n_{1,1}} & V_{n_{1,2}} \\
\vdots & \vdots \\
V_{n_{N,1}} & V_{n_{N,2}}
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\vdots \\
\gamma_N \\
\gamma_{N+1}
\end{pmatrix}
\]
**Step 5: Adding the Freestream Velocity \( \rightarrow \) RHS**

\[
\begin{pmatrix}
V_{\text{inf}} \cdot n_1 \\
V_{\text{inf}} \cdot n_2 \\
V_{\text{inf}} \cdot n_3 \\
V_{\text{inf}} \cdot n_4 \\
V_{\text{inf}} \cdot n_5 \\
\vdots \\
V_{\text{inf}} \cdot n_N
\end{pmatrix}
\begin{pmatrix}
V_{n,1,1} & V_{n,1,2} & V_{n,1,3} & \cdots & V_{n,1,N+1} \\
V_{n,2,1} & V_{n,2,2} & V_{n,2,3} & \cdots & V_{n,2,N+1} \\
V_{n,3,1} & V_{n,3,2} & V_{n,3,3} & \cdots & V_{n,3,N+1} \\
V_{n,4,1} & V_{n,4,2} & V_{n,4,3} & \cdots & V_{n,4,N+1} \\
V_{n,5,1} & V_{n,5,2} & V_{n,5,3} & \cdots & V_{n,5,N+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_{n,N,1} & V_{n,N,2} & V_{n,N,3} & \cdots & V_{n,N,N+1}
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\gamma_N \\
\gamma_{N+1}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{pmatrix}
\]

\[
\overrightarrow{V}_\infty \cdot \hat{n}_j + \sum_{i}^{NP} \int_{x'} \hat{n}_j \left( \Delta \overrightarrow{v}_\gamma(x') \right) dx' = 0
\]
Step 5: Adding the Freestream Velocity → RHS

\[
\begin{bmatrix}
V_{\text{inf}} \cdot n_1
V_{\text{inf}} \cdot n_2
V_{\text{inf}} \cdot n_3
V_{\text{inf}} \cdot n_4
V_{\text{inf}} \cdot n_N
\end{bmatrix}
\begin{bmatrix}
V_{n,1,1}
V_{n,2,1}
V_{n,3,1}
V_{n,4,1}
V_{n,5,1}
\vdots
V_{n,N,1}
\end{bmatrix}
+ \begin{bmatrix}
\gamma_1
\gamma_2
\gamma_3
\gamma_4
\gamma_5
\gamma_6
\gamma_N
\gamma_{N+1}
\end{bmatrix}
= \begin{bmatrix}
0
0
0
0
0
0
\end{bmatrix}
\]
**Step 5:** Adding the Freestream Velocity $\rightarrow$ RHS

\[
\begin{pmatrix}
V_{n1,1} & V_{n1,2} & V_{n1,3} & \cdots & V_{n1,N+1} \\
V_{n2,1} & V_{n2,2} & V_{n2,3} & \cdots & V_{n2,N+1} \\
V_{n3,1} & V_{n3,2} & V_{n3,3} & \cdots & V_{n3,N+1} \\
V_{n4,1} & V_{n4,2} & V_{n4,3} & \cdots & V_{n4,N+1} \\
V_{n5,1} & V_{n5,2} & V_{n5,3} & \cdots & V_{n5,N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
V_{nN,1} & V_{nN,2} & V_{nN,3} & \cdots & V_{nN,N+1}
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\gamma_N \\
\gamma_{N+1}
\end{pmatrix}
= 
\begin{pmatrix}
-V_{\text{inf}} \cdot n_1 \\
-V_{\text{inf}} \cdot n_2 \\
-V_{\text{inf}} \cdot n_3 \\
-V_{\text{inf}} \cdot n_4 \\
-V_{\text{inf}} \cdot n_5 \\
\vdots \\
-V_{\text{inf}} \cdot n_N \\
\end{pmatrix}
\]
Some Possible Solutions:

No Net Circulation

Net Negative Circulation

Net Positive Circulation

So, which solution is the correct one, and how do we ensure we get that solution?
Smooth T.E. Flow
What Should Happen at the Trailing Edge?

- We can enforce pressure of upper surface and lower surface flows to be the same at the trailing edge.

- For steady flows, pressure equality at trailing edge implies velocity equality at trailing edge (From Bernoulli Equation).

\[ \frac{1}{2} \rho V^2 = \frac{1}{2} \rho V^2 \]

\[ V^2 = V^2 \]

\[ |V| = |V| \]
Added Comment on Velocities at the Trailing Edge

• For a finite angled trailing edge:

• For a cusped trailing edge:

See Anderson for details.
Condition which is applied in the panel method: The Kutta Condition

- We should implement a Kutta Condition into the linear vortex panel method to ensure there is no net vorticity at the T.E.
Step 6: Implementing the Kutta Condition in the Linear Vorticity Panel Method

\[ \begin{align*}
\gamma_1 + \gamma_{NP+1} &= 0
\end{align*} \]
Step 7: Solve the System of Equations

\[
\begin{pmatrix}
V_{n,1}^{1,1} & V_{n,1}^{1,2} & V_{n,1}^{1,3} & \cdots & V_{n,1}^{1,N+1} \\
V_{n,1}^{2,1} & V_{n,1}^{2,2} & V_{n,1}^{2,3} & \cdots & V_{n,1}^{2,N+1} \\
V_{n,1}^{3,1} & V_{n,1}^{3,2} & V_{n,1}^{3,3} & \cdots & V_{n,1}^{3,N+1} \\
V_{n,1}^{4,1} & V_{n,1}^{4,2} & V_{n,1}^{4,3} & \cdots & V_{n,1}^{4,N+1} \\
V_{n,1}^{5,1} & V_{n,1}^{5,2} & V_{n,1}^{5,3} & \cdots & V_{n,1}^{5,N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
V_{n,1}^{N,1} & V_{n,1}^{N,2} & V_{n,1}^{N,3} & \cdots & V_{n,1}^{N,N+1} \\
1 & 0 & 0 & \cdots & 1
\end{pmatrix} \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\gamma_N \\
\gamma_{N+1}
\end{pmatrix} = \begin{pmatrix}
-V_{\text{inf}} \cdot n_1 \\
-V_{\text{inf}} \cdot n_2 \\
-V_{\text{inf}} \cdot n_3 \\
-V_{\text{inf}} \cdot n_4 \\
-V_{\text{inf}} \cdot n_5 \\
\vdots \\
-V_{\text{inf}} \cdot n_N \\
0
\end{pmatrix}
\]
Step 8: Postprocessing the Results

- Circulation:

\[ \Gamma = \sum_{j=1}^{NP} \bar{\gamma}_j s_j \]

- Lift:

\[ L' = \rho V_\infty \sum_{j=1}^{NP} \bar{\gamma}_j s_j \quad \text{(Kutta Joukowski)} \]

- Pressure Coefficient:

\[ C_p = \frac{p - P_\infty}{1/2 \rho V_\infty^2} = 1 - \frac{v_t^2}{V_\infty^2} \]
Summary of Key Steps (These steps apply to most panel methods)

• Get the geometry
• Place panels onto the geometry
• Choose a panel type (source, vortex, source-doublet, etc.)
• Setup the Aerodynamic Influence Coefficient matrix
• Generate a RHS
• Add a Kutta condition
• Solve the system of equations
• Post-process the result
Pressure Coefficient

![Graph showing Pressure Coefficient](image)

- Upper Surface
- Lower Surface

Mesh Vertices
Panel Centroids
-1$^\circ$C$_p$
The Source Panel Method

- Source Panel Method:

\[ \hat{n} \cdot \nabla (\phi(r)) = \hat{n} \cdot \nabla \left( -\frac{1}{2\pi} \int_{S} \sigma \ln(r) dl \right) \]

- **NOTE:** The source-only-panel method can **not** be used to compute the flow around a lifting airfoil. Unless modified, the source panel method will always solve for the zero-circulation flow around a body.
Source-Doublet Panel Method

• Source-Doublet Panel methods

\[
\phi(r) = \frac{1}{2\pi} \left( \int_{S_\mu} \mu \frac{\partial}{\partial n}(ln(r))dl - \int_{S_\sigma} \sigma ln(r)dl \right)
\]

• Source doublet panel methods are commonly used and can be used to solve for the flow around lifting bodies.
Other Options for Panel Methods

• Source-Doublet Panel methods

\[
0 = \frac{1}{2\pi} \left( \int_{S_\mu} \mu \frac{\partial}{\partial n}(ln(r))dl - \int_{S_\sigma} \sigma \ln(r)dl \right)
\]

• Source doublet panel methods are commonly used in 3D
• They can be used to solve for the flow around lifting bodies.
Applying Panel Methods to Modern 3D Problems
Trailing Vortices
Trailing Vortices

Cessna airplane. Photo by Paul Bowen.
From a Spitfire to Lifting Line Model.

Determining the prescribed bat geometry

- Typical flight conditions:
  - Forward velocity 2-7 m/s
  - 7 cm chord
  - 40 cm wingspan
  - Flapping frequency 8-10 Hz

- Kinematics:
  - Synchro-Stereo imaging at 1000 fps

- Wake velocities:
Wingbeat Re-Constructions

Front View

Side View

Top View
Example of Similar Methods in Action
Develop a mesh/discretization and Apply a Vortex Distribution

On each of the panels we place a distribution of vorticity
Setup a Linear Matrix System

• The matrix system is $N_V \times N_V$ in size:
  • For large number of “panels” or “elements” it is costly to solve this matrix system.

• We apply techniques which approximate the matrix system, allowing us to rapidly solve the problem.
Unsteady Effects: From a Spitfire to a flapping wing.

Flapping wing
- Unsteady motions
- Unsteady wake structures
Flapper Design: 3-D Structural Tailoring

Typical Level Flight Result

Outboard Section  
Lower Stiffness A-A

Inboard Section  
Higher Stiffness B-B

MOVIE
Wing Design Example

Select Wing Planform → Determine Feathered 3D Wing Shape → Determine Shape Modification
Results…