

Radio Riot

Introduction

In this paper we present our solution to the 2011 MCM problem 'B'. The problem pertains to finding the minimum number of very high frequency (VHF) radio repeaters to service 1,000 or 10,000 simultaneous users in a 40-mile radius. We have extended this to include the placement of those repeaters which we believe to be an equally important issue. We present a solution to the problem, discuss the effects of mountainous terrain, and also describe some alternative approaches.

Assumptions

'Users' of radio may consist of both transmitting users and receiving users. Users transmitting may have any number of receivers (even none) listening. For the purpose of answering this problem, we have chosen to assume that every transmission is being received by exactly one user since this represents the worst possible load on the repeater network without having any 'worthless' (not being received) transmissions. This means there is exactly one transmission for every pair of users. We also assume that the problem's use of the word 'simultaneous' implies that every pair of users is trying to transmit at the same time and must have access to an open repeater.

Our research on how repeaters work has led us to believe that repeaters can only be tuned to accept one frequency as well as transmit one frequency. This trivializes the problem of finding the minimum number of repeaters required to service any given population of users. Working on our previous assumption that every transmission can be matched to one pair of users, the minimum number of required repeaters must, in general, equal to the number of conversations we wish to sustain, or half the number of users.

Frequency spacing between radio transmission frequencies must be adequately large to avoid interference. Based on information provided by the Utah VHF Society [1] we chose a minimum spacing of 24kHz (slightly larger than recommended minimum) between frequencies. We also follow their recommendation of leaving a gap of at least 10kHz on each edge of the band to prevent bleed over into adjacent bands.

Developing the Model

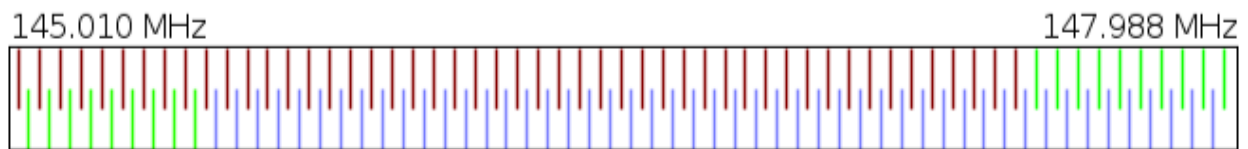
Partitioning the Spectrum

The frequency spacing was determined using a naive approach that assigned 49 repeater inputs at 48 kHz intervals. We chose this number because it allowed us to place repeaters such

that, if all repeaters transmit at 600 kHz higher than the signal they receive, all input and output channels have a spacing of at least 24 kHz. We start the inputs at 145.010 MHz to avoid the possibility of a signal bleeding out below 145 MHz, with the assumption that the VHF transmissions will have a bandwidth of around 16 kHz [1]. The 600 kHz band from 145.010 MHz to 145.610 MHz will not have any repeater outputs, so in a distribution like this one could place simplex channels, which only transmit data, between the repeater inputs to allow more flexibility. This could also be done in the top portion of the spectrum where only repeater output frequencies occur, resulting in a total of 19 simplex frequencies.

Repeater inputs occur at: $145.010 + 0.048k \text{ MHz}$, for k from 0 to 49.

Repeater outputs occur at: $145.610 + 0.048k \text{ MHz}$, for k from 0 to 49.



Each frequency is separated by 24 kHz. The dark red (dark grey) lines represent repeater input frequencies, light blue (light grey) repeater output frequencies. The green (grey) frequencies represent possible simplex frequencies.

Using the 49 frequencies yielded by this partition and the 54 available PL Tones, one can create 2,646 unique frequency/PL Tone pairs henceforth referred to as 'channels'. All of these channels may simultaneously carry signals in the same airspace.

Transmitter Range

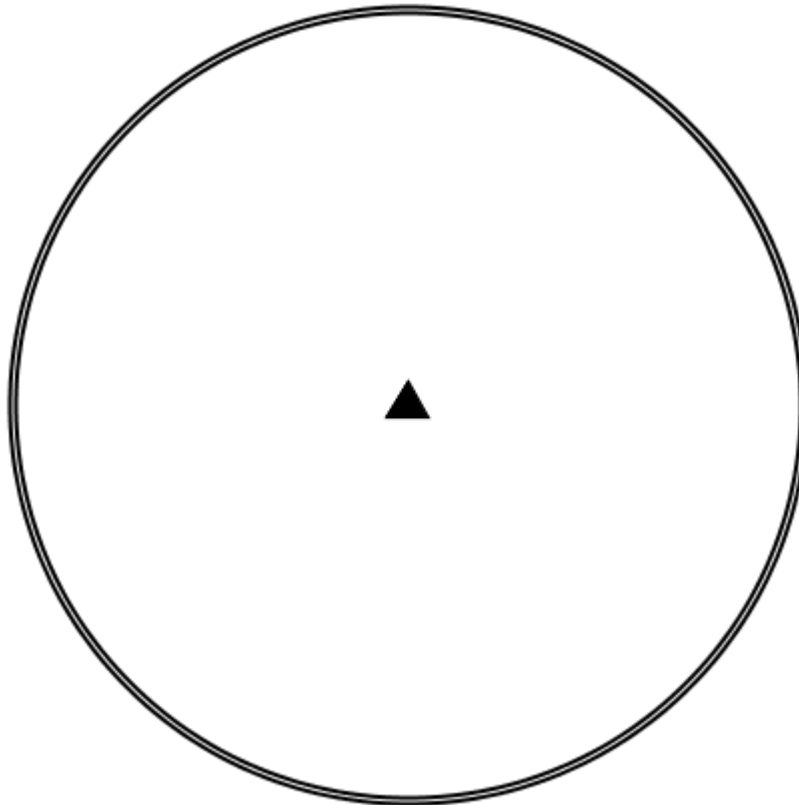
Since repeaters have a line-of-sight broadcasting range, we can determine the height of an antenna that is required for the transmitter ranges demanded by the model. Using simple geometry, a few approximations, and some dimensional analysis [5], we can say that

$$d[\text{miles}] \approx \sqrt{1.5 * h[\text{feet}]}$$

where d is the transmitter range in miles and h is the required antenna height in feet. When we refer to antenna height in this paper, we use this equation to calculate the given values.

The Simple Case: 1,000 Simultaneous Users

Given the assumption that users operate in pairs given above, accommodating 1,000 users simultaneously is a matter of having 500 total channels being serviced by repeaters. Because we have over 500 channels available in any given airspace, this can be accomplished by simply placing 500 repeaters in the geographic center of the region, each servicing a different channel over the entire region. See the figure below for a basic representation.



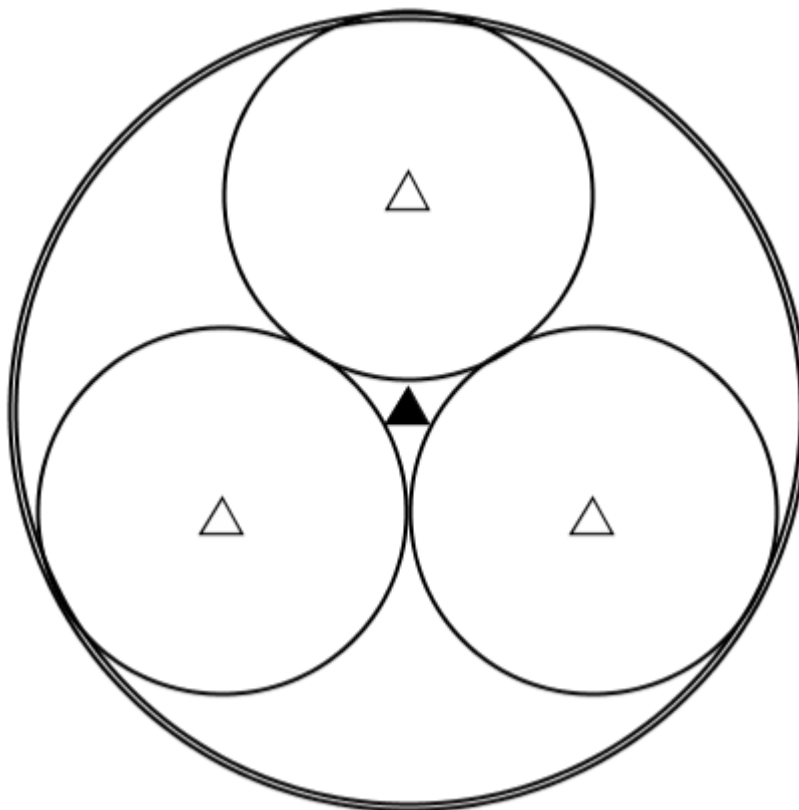
The black triangle represents a cluster of 500 repeaters with a 40-mile range (height of about 1,067ft).

10,000 Simultaneous Users

Supporting 10,000 simultaneous users becomes more complicated than supporting 1,000. 10,000 users means 5,000 channels available, and since we only have 2,646 channels available we must artificially create more through geographic separation. We consider the range of each transmission to be a worthy concern (though secondary to supporting the required number of users with the fewest repeaters), so we chose to partition the channels into two sets. One set is operated region-wide from the geographic center while the other set is used to service three smaller circular areas. By servicing 1,469 channels using the center station and 1,177 with each of the smaller ones we achieve exactly 5,000 channels and thus 5,000 simultaneous users. There are two clear solutions to placing the secondary stations depending on policy regarding neighboring areas discussed below.

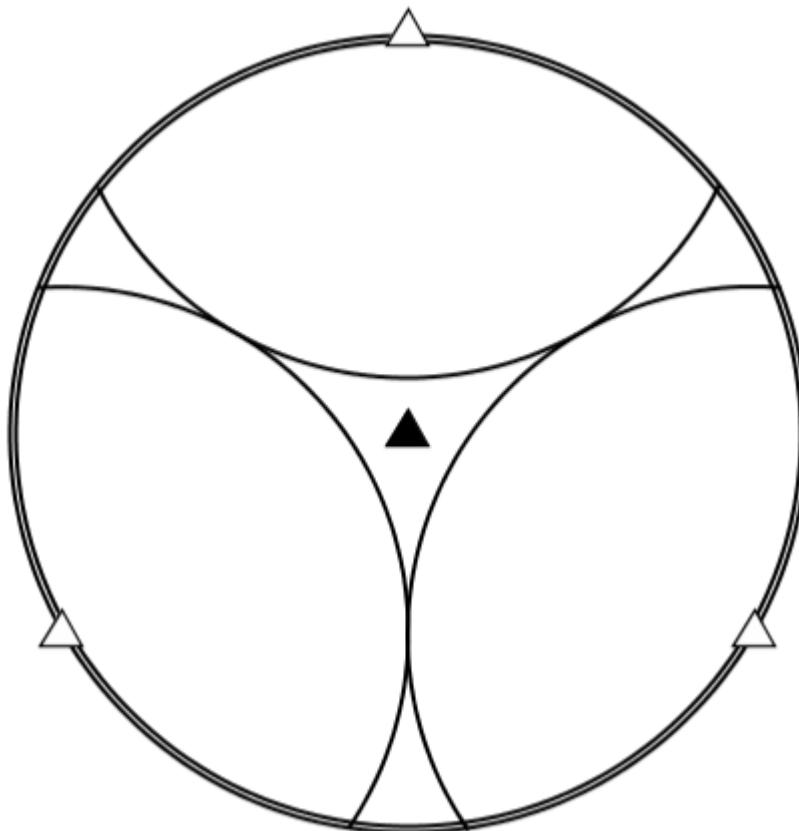
In the first layout, we consider that neighboring areas may have their own requirements for VHF transmissions that our repeaters would possibly interfere with if their broadcast radii fall outside our 40-mile radius area. In this case, we want to address two issues. First, we must make sure that no repeater is placed so that it broadcasts outside our jurisdiction. Second, we want to maximize the distance over which someone can transmit signals using this secondary repeater network. This requirement is in direct opposition to the problem of maximising the total area this repeater network covers, but the fact that we still have the central cluster to fall back on makes

this easier to deal with. As such, we suggest the following configuration which is comprised of three secondary repeater clusters which together cover approximately 64.6% of the total area.



The black triangle represents a cluster of 1,469 repeaters with a 40-mile range (height of about 1,067ft). The white triangles represent clusters operating the remaining 1,177 channels at a range of 17.32 miles (height of about 200ft).

The second situation in which we do not have to worry about interfering with neighboring regions' VHF transmissions is the one we consider more agreeable. In this scenario, we can place our repeaters on the border of the circle, thus giving us more options for increasing both the coverage and the maximum broadcasting distance of the secondary repeater network. For this case, we propose placing three repeater clusters, spaced by 120 degrees, on the border of the circle. This configuration allows for approximately 82.7% coverage for the secondary network, which is a vast improvement over the previous scenario.



The black triangle represents a cluster of 1,469 repeaters with a 40-mile range (height of about 1,067ft). The white triangles represent clusters operating the remaining 1,177 channels at a range of 34.64 miles (height of about 800ft).

Testing/Sensitivity

This model was developed with the intent to spread coverage fairly uniformly over the given area, and thus it does not consider population density. By using a function $P(x,y)$ which assigns a probability that a user will require a repeater at any given coordinate in the plane, we can compare coverage to population density and test for deviations. The function P is explained further in the section 'Considering Population Density' below.

The function P can also be used for discrete testing by generating random user pairs that wish to communicate. Each pair can be assigned a repeater if available. By generating an appropriately large population of users one can determine how many of those users can successfully communicate at the same time.

As it stands, our model shows little sensitivity to population other than between the 1,000 users and 10,000 users cases. In general, each of the above approaches is applicable to a large number of possible users before a new approach is required. For example, given that we have 2,646 available channels, the 1,000 user approach works for up to 5,292 users. The only

difference between models covering different population sizes is the number of repeaters in the central cluster.

Mountainous Broadcast Considerations

In order to deal with the issue of line-of-sight obstructions such as mountains, our model will have to be adjusted accordingly. An easy solution would seem to be placing repeater clusters on the top of the highest peak in the region. The increased elevation would minimize the loss of coverage by elevating the antenna well above much of the obstruction. However, the elevation increase will greatly increase the path along which the signal will travel to reach its destination, thus increasing the likelihood of atmospheric refraction [6].

An approach that was left out of the above model was chaining repeaters. This was not included for the reason that the repeaters in the chain would have to be calibrated to each other and would therefore only increase the number of repeaters used to service the same number of users. In this case however, repeater chains could be utilized to service smaller populations by re-routing radio traffic around major obstructions. This would require careful planning, but it is not out of the question. It is important to note that our choice for the frequency spacing, with repeaters only shifting the frequency up as opposed to down, would be helpful in designing repeater chains because it eliminates the possibility of a repeater pair being caught in a loop.

Strengths & Weaknesses

Strengths

Circle packing, which is used in this model to deal with situations where a single repeater cluster is insufficient, is a well understood geometric problem [4]. This means extending the solution to larger populations should be relatively trivial. In general, two overlapping circle packings operating different channels sets should be an appropriate solution for any large number of simultaneous users.

In our research, we were unable to confirm that one antenna/mounting can service more than one repeater, but if this is the case then our cluster approach is a very cost-effective solution to covering the area.

Simplex communication is radio communication directly from transceiver to transceiver without the use of a repeater. While this can drastically limit the range of transmission, it allows for communication both in the absence of a repeater and if using a repeater is undesirable. Our band partitioning allows for a number of simplex frequencies for the benefit of users who need or desire this form of communication.

Unlike some more complex methods, such as those we describe later, our approach requires little to no additional computation. Application of our model should be fairly straight-forward assuming no environmental considerations are necessary.

Weaknesses

As was mentioned above, we are not certain if a single antenna/mounting can service multiple repeaters, but if this is not the case, then there will be serious issues in terms of separating hundreds of antennas enough without interfering too much with their transmission range. Additionally, the cost of constructing hundreds or thousands of antennas to service the clusters would be enormous.

Many amateur radio users tend to believe that PL tones do not assist in the avoidance of interference issues. One source claims that the number of PL tones and the total number of channels in any given area are entirely independent of one another[3]. The Delaware Repeater Association explains how signals of the same frequency create “beat notes” regardless of the use of PL tones[2]. Should this information be true, our approach could be subject to significant interference issues.

Our method tends to use more long-range repeaters than some of the others we mention below. While these repeaters do provide more coverage, the installation and maintenance costs may exceed those of other approaches.

As was discussed in the section on mountainous obstructions, this model is dependent on the assumption of a flat area to distribute repeaters. The difficulty of adjusting this model to fit a specific environment scales with the complexity of obstructions, and we were unable to work some of our approaches to dealing with these issues into the final solution.

Other Approaches

The model proposed above is very simplistic, and thus prone to difficulties when adjusting to specific user distributions. In our time working towards a solution, we developed additional methods of placing repeaters that we were unable to fully integrate into our model. This work would greatly improve our model and fix many of the problems with it, but it is currently incomplete.

Considering Population Density

A good model would be general enough to support a wide variety of population distributions. In order to address the limitless possibilities for user distributions, we propose a probabilistic model that will predict the likelihood of a user occupying a location based off of their proximity to landmarks such as major cities or towns. The probability distribution we use is a modification of Newton’s Law of Gravitation, where cities and towns are treated as massive bodies, and users are treated as objects whose masses are comparatively insignificant. We first use the following function to describe the magnitude of ‘force’ on a user from a landmark:

$$F(x,y) = \frac{S}{\Delta x^2 + \Delta y^2}$$

where S is the 'mass' or size of the city, (x,y) is the location of the user, and $\Delta x, \Delta y$ describe the rectangular displacement of the user from the center of the landmark. This assigns a value to a position that is proportional to the inverse square of the displacement. Notice that S will determine not only the magnitude of the force, but it is also capable of describing a repulsion by giving it a negative value. This could be used in the case of landmarks such as landfills, where the number of users is likely to be much smaller in the immediate vicinity. We suggest that negative S values be used sparingly, or at least carefully, so as to avoid a negative probability result in the final equation.

Since there could be many landmarks, we generalize the total force from k landmarks as:

$$F_{Total}(x,y) = \sum_{all k} F_k(x,y) = \sum_{all k} \frac{S_k}{\Delta x_k^2 + \Delta y_k^2}$$

Now we can say that the total force, or weight, of the entire area is given by:

$$W = \sum_{all k} S_k \int_0^{2\pi} \int_0^{R_k(\theta)} \frac{1}{r_k} dr_k d\theta$$

If the origin is placed at the center of the circle, then $R_k(\theta)$ is the distance from the center of landmark k to the point on the circumference at angle θ from the origin. The integrals are expressed in polar coordinates because they are less cluttered in this form. It is worth noting that we were unable to solve this equation, but knowing that an answer exists, we can finally declare the probability distribution of users as follows:

$$P(x,y) = \frac{1}{W} F_{Total}(x,y) = \frac{1}{W} \sum_{all k} \frac{S_k}{\Delta x_k^2 + \Delta y_k^2}$$

where P gives us a number between 0 and 1. Now that we have a method of predicting the likelihood that any given location requires the service of a repeater, they can be placed accordingly. This approach works nicely with the following method.

Imposing A Grid

Another model we came up with involved a recursive process of dividing the service area into four equal area partitions and placing a number of repeaters, each with a range spanning the entire area of their respective partition, based on the number of users in each partition. First, we consider a square which inscribes the entire service area and place an appropriate number of

40-mile range repeaters in the center of this square. Next, we divide the square into four equal-area partitions, analyze the population of each partition, and place shorter range repeaters in the centers of each partition. This process of dividing the partitions and placing repeaters continues until the total number of necessary repeaters is reached. Assigning repeaters to accept frequencies based on their range will prevent interference between overlapping repeaters (e.g. ones which cover partitions of different sizes). Choosing the transmission range of repeaters in a given partition such that their boundaries form a circle which is inscribed in that partition causes repeaters of similar range to form a planar graph. Thus, by the four-color theorem, we should only need four PL tones to ensure no interference between repeaters [7].

Strengths

- Saves money spent on repeater installation and maintenance by using short-range repeaters in place of long-range ones
- Practical, intuitive distribution; populated areas get more repeaters

Weaknesses

- Does not describe how many repeaters to distribute given some population.
- Will not ensure that every person can talk to whomever they wish to at any given time as long-range repeaters may already be in use
- Choosing repeater ranges such that their boundaries are inscribed in their respective partitions prevents them from providing coverage to the entire partition. In order to service those users, the range must be chosen such that the repeater's boundary circumscribes the square. Thus, the signal could cause interference problems with neighboring repeaters, and more PL tones will be required to prevent them.

Iterative Algorithms

If we assume that every user needs their own repeater, we can find a very good estimate of the best placement of each repeater by using an iterative algorithm. Given each user's position, their maximum transmission range, and their repeater's maximum transmission range we can create an algorithm to find the location at which to place the repeater such that it maximizes the number of users within its range.

Strengths

- Can be used with repeaters of any range
- Tells exactly where to put repeaters so they reach the most users
- Allows each user to have the highest probability (given their repeater's range) of being able to contact the person they wish to

Weaknesses

- Can take a long time to compute for large input
- Does not explain how to distribute frequencies and PL tones
- Can only give an approximate answer of the best layout of repeaters

- Assumes that each user has their own repeater. This method can be modified to output a sampling of repeater locations, but such a modification could yield results which leave users outside of the range of any repeaters.

References:

[1] <http://utahvhfs.org/bandplan1.html>

[2] <http://www.dra73.org/pl.html>

[3] http://www.museumstuff.com/learn/topics/PL_tone::sub::Uses

[4] <http://mathworld.wolfram.com/CirclePacking.html>

[5] <http://cnx.org/content/m0538/latest/>

[6] Richards, John A., Radio Wave Propagation: An Introduction for the Non-Specialist, Springer Verlag, Berlin. 2008.

[7] <http://mathworld.wolfram.com/Four-ColorTheorem.html>