Exam 2 Solution

1. Since we can recognize that

$$f_Y(y) = cy^2/e^y = cy^2e^{-y} = cy^{3-1}e^{-y/1}$$

must be a gamma pdf with $\alpha = 3$ and $\beta = 1$, we can see that

$$f_Y(y) = \frac{y^{3-1}e^{-y/1}}{\Gamma(3)1^3} = \frac{y^{3-1}e^{-y/1}}{(3-1)!\cdot 1} = \frac{y^{3-1}e^{-y/1}}{2}$$

from which we immediately conclude that

$$c = 1/2.$$

2. Since, $f_{Y_1}(y_1) = \int_{y_1}^{\infty} e^{-y_2}\,dy_2 = e^{-y_1}$, we have that

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f(y_1,y_2)}{f_{Y_1}(y_1)} = \frac{e^{-y_2}}{e^{-y_1}} = e^{-y_2}e^{y_1} = e^{-(y_2-y_1)}, \quad 0 < y_1 < y_2 < \infty.$$

3. (a) We have that

$$\begin{aligned}
E(Y_1Y_2) &= \iint y_1y_2 f(y_1,y_2)\,dy_1dy_2 = \int_0^1\int_0^{1-y_2} y_1y_2 24y_1y_2\,dy_1dy_2 \\
&= \int_0^1\int_0^{1-y_2} 24y_1^2y_2^2\,dy_1dy_2 = 24\int_0^1 y_2^2\int_0^{1-y_2} y_1^2\,dy_1dy_2 \\
&= 8\int_0^1 y_2^2(1-y_2)^3\,dy_2 = 8\int_0^1 y_2^{3-1}(1-y_2)^{4-1}\,dy_2 \\
&= 8\frac{\Gamma(3)\Gamma(4)}{\Gamma(3+4)}\int_0^1 \frac{\Gamma(3+4)}{\Gamma(3)\Gamma(4)}y_2^{3-1}(1-y_2)^{4-1}\,dy_2 = 8\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = 8\frac{2!3!}{6!} = \frac{2}{15}
\end{aligned}$$

(b) (Bonus) Noting that it is easier to integrate over Y_2 first and that the bound for Y_1 stops at 1/2 in this way (why?),

$$\begin{aligned}
P(Y_1 \le Y_2) &= \int_0^{1/2}\int_{y_1}^{1-y_1} 24y_1y_2\,dy_2dy_1 \\
&= \int_0^{1/2} 12y_1\left[y_2^2\Big|_{y_1}^{1-y_1}\right]dy_1 = \int_0^{1/2} 12y_1[(1-y_1)^2 - y_1^2]\,dy_1 \\
&= \int_0^{1/2} 12y_1[1-2y_1]\,dy_1 = \int_0^{1/2}(12y_1 - 24y_1^2)\,dy_1 \\
&= \left[6y_1^2 - 8y_1^3\Big|_0^{1/2}\right] = \frac{6}{4} - \frac{8}{8} = \frac{1}{2}
\end{aligned}$$