Spring 2025

Homework 6

Due Thursday, March 20

1. Let X be a random variable with probability density function

$$f(x) = c(1 - x^2), \quad -1 < x < 1$$

- (a) What is the value of c?
- (b) What is the cumulative distribution function of X?
- 2. The lifetime of a machine part has a continuous distribution on the interval (0, 40) with probability density function $f(x) = c(10 + x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 6.
- 3. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by $f(x) = cxe^{-x/2}$, x > 0, what is the probability that the system functions for at least 5 months?
- 4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \frac{10}{x^2}, \quad x > 10$$

- (a) Find P(X > 20).
- (b) What is the cumulative distribution function of X?
- (c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?
- 5. The length of time to failure (in hundreds of hours) for a transistor is a random variable Y with distribution function (cdf) given by

$$F_X(x) = 1 - e^{-x^2}, \quad x \ge 0.$$

- (a) Find the pdf of X, $f_X(x)$.
- (b) Find the probability that the transistor operates for at least 200 hours.
- (c) Find the probability that the transistor operates for more than 100 hours, given that it has already operated for 200 hours.
- 6. The density function of X is given by $f(x) = a + bx^2$, $0 \le x \le 1$. If E[X] = 3/5 find a and b.
- 7. The standard deviation of X, denoted SD(X), is given by

$$SD(X) = \sqrt{\operatorname{Var}(X)}$$

Find SD(aX + b) if X has variance σ^2 (your answer should be in terms of σ).

8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}, \quad x \ge 0.$$

Compute the expected lifetime of such a tube.

9. Weekly CPU time used by an accounting firm has a pdf given by

$$f(x) = \frac{3}{64}x^2(4-x), \quad 0 < x < 4$$

- (a) Find the expected value and variance of weekly CPU time.
- (b) The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.
- (c) Would you expect the weekly cost to exceed \$600 very often? Why?
- 10. If the pdf of X is given by f(x) = |x|/10, -2 < x < 4, find E[X].
- 11. If the pdf of X is given by $f(x) = e^{-x}$, x > 0, and if $Y = \max(X, 2)$, find E[Y]. (Hint: Recall that $\max(a, b) = a$ if $a \ge b$ and $\max(a, b) = b$ if $a \le b$).
- 12. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, X, follows a distribution with density function $f_X(x) = 2x^{-3}$, x > 1. Calculate the expected value of the benefit paid under the insurance policy. (HINT: The "benefit paid' in this case is given by Y, which is defined, along with "loss/claim" X and "benefit limit" u, in the Policy Limit section on page 3 of Exam P Formula Sheet).
- 13. (5090*) An insurance policy pays for a random loss X subject to a deductible of d, where 0 < d < 1. The loss amount is modeled as a continuous random variable with density function f(x) = 2x, 0 < x < 1. Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64. Calculate d. (HINT: See again the Insurance Deductible section on page 3 of Exam P Formula Sheet).
- 14. (5090*) If X is a continuous random variable, show that $E[(X a)^2]$ is minimized when a = E[X].
- 15. (5090*) If T is a continuous random variable with pdf $f_T(t)$ and cdf $F_T(t)$, show that

$$\lim_{\delta \to 0} \frac{P(t \le T < t + \delta | T \ge t)}{\delta} = \frac{f_T(t)}{1 - F_T(t)} = -\frac{d}{dt} \log(1 - F_T(t))$$