Homework 6
Due Thursday, March 14

1. Consider a random variable Y with the probability functions

$$P(Y=1)=0.4, \quad P(Y=2)=0.3, \quad P(Y=3)=0.2, \quad P(Y=4)=0.1$$

Specify the value of the distribution function (cdf) $F_Y(y)$ and sketch.

2. A box contains five keys, only one of which will open a lock. Keys are randomly selected and tried, one at a time, until the lock is opened (keys that do not work are discarded before another is tried). Let Y be the number of the trial on which the lock is opened.
   (a) Find the probability function for Y
   (b) Give the corresponding distribution function (cdf)
   (c) What is $P(Y<3)$? $P(Y \leq 3)$? $P(Y=3)$?

3. Let Y be a random variable with probability density function $f_Y(y)=c(1-y^2)$, $-1<y<1$.
   (a) What is the value of constant c?
   (b) What is the cdf of Y?

4. The probability density function (pdf) of Y, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f_Y(y)=\frac{10}{y^2}, \quad y>10.$$

   (a) Find $P(Y>20)$.
   (b) What is the cumulative distribution function of Y?
   (c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours?

5. The length of time to failure (in hundreds of hours) for a transistor is a random variable Y with distribution function given by

$$F_Y(y)=1-e^{-y^2}, \quad y \geq 0.$$

   (a) Find the 30-th quantile (0.30-quantile) of Y.
   (b) Find the pdf of Y, $f_Y(y)$.
   (c) Find the probability that the transistor operates for at least 200 hours.
   (d) Find the probability that the transistor operates for more than 100 hours, given that it has already operated for 200 hours.

6. The density function of Y is given by $f_Y(y) = a + by^2$, $0 \leq y \leq 1$. If $E(Y) = 3/5$, find the constants a and b. (Hint: Use $\int f_Y(y)\,dy = 1$).

7. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f_Y(y) = ye^{-y}, \quad y \geq 0.$$

Compute the expected lifetime of such a tube.

8. Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f_Y(y) = \frac{3}{64}y^2(4-y), \quad 0 < y < 4$$

   (a) Find the expected value and variance of weekly CPU time.

   (b) The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

   (c) Would you expect the weekly cost to exceed \$600 very often? Why?

9. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function $f_Y(y) = 2y^{-3}$, $y > 1$. Calculate the expected value of the benefit paid under the insurance policy.

10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M. If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he/she go to destination A? (Assume that he/she cannot catch the train when arrived exactly at 7.)

11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than 1/4.

12. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

    (a) What is the probability that you will have to wait longer than 10 minutes?

    (b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

13. The median of the distribution of a continuous random variable Y is the value m such that $P(Y \leq m) = 0.5$. What is the median of the uniform distribution on the interval (θ_1, θ_2)?

14. If Y is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

    (a) $P(Y > 5)$

    (b) $P(4 < Y < 16)$

15. The annual rainfall (in inches) in a certain region is normally distributed with mean of 40 and standard deviation of 4. What is the probability that, starting with this year, we will have 10 years in a row of a rainfall of 50 inches or less? What assumptions are you making?

16. **(5090*)** If Y is a nonnegative continuous random variable (i.e., $Y \geq 0$), show that

$$E(Y) = \int_0^\infty P(Y > y)\, dy$$

17. **(5090*)** If Y is a continuous random variable, show that $E[(Y - a)^2]$ is minimized when $a = E(Y)$.

18. **(5090*)** If Y is a continuous random variable with pdf $f_Y(y)$ and cdf $F_Y(y)$, show that

$$\lim_{\delta \to 0} \frac{P(y \leq Y < y + \delta | Y \geq y)}{\delta} = \frac{f_Y(y)}{1 - F_Y(y)} = -\frac{d}{dy} \ln(1 - F_Y(y))$$