Homework 7

Due Thursday, March 27

- 1. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M. If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A? (Assume that he cannot catch the train when arrived exactly at 7).
- 2. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
 - (a) What is the probability that you will have to wait longer than 10 minutes?
 - (b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?
- 3. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than 1/4.
- 4. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute
 - (a) P(X > 5)
 - (b) P(4 < X < 16)
- 5. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, we will have 10 years in a row of a rainfall of 50 inches or less? What assumptions are you making?
- 6. In a town, it was found that out of one out of every six people is insured. Consider a random sample of 612 people from the town. What is the probability that the number of insured people is strictly between 90 and 150, using a normal approximation with the continuity correction?
- 7. An airline finds that 5% of the persons who make reservations on a certain flight do not show up for the flight. If the airline sells 160 tickets for a flight, what is the probability that there are at most 5 no shows on a flight, using
 - (a) Exact binomial probability?
 - (b) Poisson approximation to binomial?
 - (c) Normal approximation to binomial with the continuity correction?

- 8. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is
 - (a) the probability that a repair time exceeds 2 hours?
 - (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?
- 9. Jones figures that the total number of miles (in thousands of miles) that an auto can be driven before it would need to be junked is an exponential random variable with mean 20. Smith has a used car that has 10,000 miles on it. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed, but rather is (in thousands of miles) uniformly distributed over (0, 40).
- 10. The number of years a radio functions is exponentially distributed with parameter $\lambda = 1/8$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?
- 11. The *median* of a continuous random variable having distribution function F is the value m such that F(m) = 1/2. That is, a random variable is just as likely to be larger than its median as it is to be smaller. To find the median, we set F(m) = 1/2 and solve for m. Find the median of X, if X is
 - (a) uniformly distributed over (a, b)
 - (b) exponential with rate λ

(Hint: see textbook for the form of cdf "F(a)" for both uniform and exponential distributions.)

- 12. The quantile (or percentile) of a random variable X, denoted by ϕ_p , where $0 \le p \le 1$, is defined as $F_X(\phi_p) = p$ (note that the 50th quantile, $\phi_{0.5}$, is the median since $F_X(\phi_{0.5}) = 0.5 = 1/2$). If the cdf of X is given by $F_X(x) = 1 - e^{-x^2}$, $x \ge 0$, find the 30th quantile of X.
- 13. The working lifetime, in years, of a particular model of client is normally distributed with mean 10 and variance 4. Calculate the 14th percentile of the working lifetime, in years.
- 14. The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. Calculate the portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year.
- 15. The monthly commission that an agent earns is modeled by an exponential random variable with mean 20. Calculate the probability that the commission the agent earns in a month is within one half of the standard deviations from the mean.
- 16. If X has $\text{Gamma}(\alpha, \lambda)$ distribution, show that, for any integer k,

$$E[X^k] = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\lambda^k}$$

- 17. If X has an exponential random variable with mean $1/\lambda$, show that $E[X^k] = k!/\lambda^k$ for any k = 1, 2, ...
- 18. Suppose that a random variable X has a pdf given by $f_X(x) = cx^3 e^{-x/2}$, x > 0. Find the value of c and conclude that X follows a χ^2 distribution (and find the appropriate degrees of freedom).
- 19. Suppose that a random variable X has a pdf given by $f_X(x) = cx^3(1-x)^2$, 0 < x < 1. Find the value of c.
- 20. The percentage of impurities per batch in a chemical product is a random variable X with density function $f_X(x) = 12x^2(1-x), 0 < x < 1$. A batch with more than 40% impurities cannot be sold. Determine the probability that a randomly selected batch cannot be sold because of excessive impurities. Also, what is the expected percentage of impurities per batch?
- 21. If X is uniformly distributed over (-1, 1), find
 - (a) P(|X| > 1/2)
 - (b) The pdf of |X|
- 22. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$. Try both the cdf method and Theorem 7.1.
- 23. Suppose that $X \sim N(0, 1)$ (standard normal) and $Y = X^2$. Find the pdf of Y.
- 24. Let X have the pdf

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

Let $Y = e^{-X}$. Find the pdf of Y.

25. Let

$$Y = \left(\frac{X - \nu}{\alpha}\right)^{\beta}$$

Show that if X is a Weibull random variable with parameters ν , α , and β , then Y is an exponential random variable with parameter $\lambda = 1$. (HINT: it's easier to use the cdf method, and use only the cdfs of Weibull and exponential; no need to find the pdf).

- 26. (5090*) Let X be the percentage score on an actuarial exam for students who did not participate in an exam-preparation seminar; X is modeled by a uniform distribution on [a, 100], where a > 0. Let Y be the percentage score on an actuarial exam for students who did participate in an exam-preparation seminar; Y is modeled by a uniform distribution on [1.25a, 100]. It is given that $E(X^2) = 19,600/3$. Calculate the 80th quantile of Y.
- 27. (5090*) The profits of life insurance companies A and B are normally distributed with the same mean. The variance of company B's profit is 2.25 times the variance of company A's profit. The 14th percentile of company A's profit is the same as the percentile ϕ_p of company B's profit. Calculate p.

- 28. (5090*) A student takes an actuarial exam with 30 questions. The probability that the student answers a given question correctly is 0.2, independent of all other questions. The probability that the student answers more than n questions correctly is greater than 0.1. The probability that the student answers more than n + 1 questions correctly is less than 0.1. Calculate n using a normal approximation with the continuity correction.
- 29. (5090*) Using the result that $E[Y] = \int_0^\infty P(Y > t) dt$ for $Y \ge 0$, show that, if $X \ge 0$,

$$E[X^n] = \int_0^\infty nx^{n-1} P(X > x) \, dx$$

30. (5090*) A distribution with a random variable X with parameter θ is said to belong to an *exponential family* if $f_X(x)$ or P(X = x) can be written as

$$h(x)c(\theta)\exp(w(\theta)t(x))$$

For example, if $X \sim \text{Exponential}(\theta)$, then X belongs to an exponential family since

$$f_X(x) = \theta e^{-\theta x} = \theta \exp(-\theta x) = h(x)c(\theta)\exp(w(\theta)t(x))$$

where

$$h(x) = 1$$
, $c(\theta) = \theta$, $w(\theta) = -\theta$, $t(x) = x$

For another example, if $X \sim \text{Binomial}(n, \theta)$ (with n fixed), then

$$P(X = x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} = \binom{n}{x} (1 - \theta)^n \left(\frac{\theta}{1 - \theta}\right)^x$$
$$= \binom{n}{x} (1 - \theta)^n \exp\left(x \log\left(\frac{\theta}{1 - \theta}\right)\right) = \binom{n}{x} (1 - \theta)^n \exp\left(\log\left(\frac{\theta}{1 - \theta}\right)x\right)$$
$$= h(x)c(\theta) \exp(w(\theta)t(x))$$

where

$$h(x) = \binom{n}{x}, \ c(\theta) = (1 - \theta)^n, \ w(\theta) = \log\left(\frac{\theta}{1 - \theta}\right), \ t(x) = x$$

Show that each of the following belongs to an exponential family.

(a) $X \sim \text{Poisson}(\theta)$ (b) $X \sim N(\theta, 1)$ (c) $X \sim \text{Beta}(2, \theta)$