Note on Delta Method
February 27, 2024

- (Delta Method) Let Y_n be a sequence of random variables that satisfies

$$\frac{\sqrt{n}(Y_n - \theta)}{\eta} \xrightarrow{d} N(0,1) \quad \text{or equivalently,} \quad \sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0,\eta^2)$$

If there is a function g such that $g'(\theta) \neq 0$, then

$$\sqrt{n}(g(Y_n) - g(\theta)) \xrightarrow{d} N(0, \eta^2[g'(\theta)]^2)$$

  - Remark: It is maybe easier to let $\theta = \mu$ and $\eta = \sigma$, so that the result looks like

$$\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0,\sigma^2) \implies \sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2)$$

  - Partial proof: Consider the Taylor series expansion of $g(Y_n)$ about θ, given by

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + R_n$$

    where the R_n is the remainder term such that $\sqrt{n}R_n \xrightarrow{p} 0$ as $n \to \infty$. Multiplying both sides by $\sqrt{n}$ and rearranging terms, we have

$$\sqrt{n}(g(Y_n) - g(\theta)) = g'(\theta)\sqrt{n}(Y_n - \theta) + \sqrt{n}R_n$$

    Apply $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0,\eta^2)$ along with $\sqrt{n}R_n \xrightarrow{p} 0$ to obtain the desired result.

  - For example, if $Y_1, \ldots, Y_n$ are iid with random variables with $E(Y_i) = \mu$ and $\text{Var}(Y_i) = \sigma^2 < \infty$, then by CLT, $\sqrt{n}(\overline{Y}_n - \mu) \xrightarrow{d} N(0,\sigma^2)$. If we want the limiting distribution of $\sqrt{n}(\overline{Y}_n^2 - \mu^2)$, then we let $g(\mu) = \mu^2$ and use the Delta Method to conclude that

$$\sqrt{n}(\overline{Y}_n^2 - \mu^2) = \sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2) = N(0, \sigma^2[2\mu]^2) = N(0, 4\mu^2\sigma^2)$$

- (Second-Order Delta Method) Let Y_n be a sequence of random variables that satisfies

$$\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0,\eta^2).$$

If there is a function g such that $g'(\theta) = 0$ but $g'(\theta) \neq 0$, then

$$n[g(Y_n) - g(\theta)] \xrightarrow{d} \eta^2 g''(\theta)\chi^2(1)/2$$

  - Partial proof: Expand the Taylor series with an additional term,

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + \frac{g''(\theta)}{2}(Y_n - \theta)^2 + R_n^*$$

where the R_n^* is the remainder term such that $nR_n^* \xrightarrow{p} 0$ as $n \to \infty$. Since $g'(\theta) = 0$, multiplying by n on both sides rearranging the terms, we obtain

$$n[g(Y_n) - g(\theta)] = \frac{g''(\theta)}{2} n(Y_n - \theta)^2 + nR_n^*$$

Now, since $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \eta^2)$ by assumption, it follows that

$$\frac{\sqrt{n}(Y_n - \theta)}{\eta} \xrightarrow{d} N(0,1) \implies \frac{n(Y_n - \theta)^2}{\eta^2} \xrightarrow{d} \chi^2(1)$$

i.e., $n(Y_n - \theta)^2 \xrightarrow{d} \eta^2 \chi^2(1)$. Use this fact, along with $nR_n^* \xrightarrow{p} 0$, to obtain the desired result.

- (Multivariate Delta Method) Suppose that $\mathbf{Y}_n$ is a sequence of random vectors that satisfies

$$\sqrt{n}(\mathbf{Y}_n - \boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma})$$

Then if $\nabla g(\boldsymbol{\theta}) \neq \mathbf{0}$,

$$\sqrt{n}(g(\mathbf{Y}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} N(\mathbf{0}, [\nabla g(\boldsymbol{\theta}))]^T \boldsymbol{\Sigma} [\nabla g(\boldsymbol{\theta})])$$