Exam 1 Solution

1. Since $U = (Y+3)/9$, it follows that $Y = 9U - 3 = 3(3U-1)$ and $\frac{dy}{du} = 9$. Hence,

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = \frac{y^2}{81}\cdot|9| = \frac{[3(3u-1)]^2}{81}\cdot 9 = \frac{9(3u-1)^2}{81}\cdot 9 = (3u-1)^2$$

for $0 < u < 1$. For $E(U)$, we compute

$$E(U) = \int u f_U(u)\,du = \int_0^1 u(3u-1)^2\,du = \int_0^1 (9u^3 - 6u^2 + u)\,du = \frac{3}{4}$$

2. Since we can write

$$f_{Y_1,Y_2}(y_1,y_2) = 2(1-y_2) = 1\cdot 2(1-y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2), \quad 0 < y_1 < 1,\ 0 < y_2 < 1$$

(can check that $f_{Y_2}(y_2)$ integrates to one), we see that

$$f_{Y_1}(y_1) = 1, \quad 0 < y_1 < 1$$

so that Y_1 is Uniform(0,1) and that

$$M_{Y_1}(t) = \frac{e^t - 1}{t}$$

3. Write

$$\begin{aligned}
\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) &= \frac{1}{n-1}\sum_{i=1}^n [(X_i - \overline{X})Y_i - (X_i - \overline{X})\overline{Y}] \\
&= \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})Y_i - \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})\overline{Y} \\
&= \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})Y_i - \frac{1}{n-1}\overline{Y}\sum_{i=1}^n (X_i - \overline{X}) \\
&= \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})Y_i
\end{aligned}$$

since $\sum_{i=1}^n (X_i - \overline{X}) = n\overline{X} - n\overline{X} = 0$.