Exam 2 Solution

1. There are many ways to solve this problem. One solution is to start with $Y \sim \text{Uniform}(\theta,\ \theta{+}1)$ which implies that $E(Y) = \theta + 1/2$ and that $\text{Var}(Y) = 1/12$. Then

$$E(\overline{Y}) = E(Y) = \theta + 1/2 \implies E(\hat{\theta}) = E(\overline{Y} - 1/2) = \theta$$

so that $\hat{\theta}$ is an unbiased estimator of θ. Hence, we can invoke Theorem 9.2, with

$$\text{Var}(\hat{\theta}) = \text{Var}(\overline{Y} - 1/2) = \text{Var}(\overline{Y}) = \frac{\text{Var}(Y)}{n} = \frac{1}{12n} \to 0$$

as $n \to \infty$, to conclude that $\hat{\theta}$ is a consistent estimator of θ.

2. First, we obtain the likelihood function

$$L(\theta) = \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n} [(1-\theta)^{y_i - 1}\theta] = (1-\theta)^{\sum_{i=1}^{n} y_i - n}\theta^n$$

then

$$\ln L(\theta) = \left(\sum_{i=1}^{n} y_i - n\right) \ln(1-\theta) + n \ln \theta$$

and setting

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{(\sum_{i=1}^{n} y_i - n)}{(1-\theta)} + \frac{n}{\theta} = 0$$

and solving for θ, we find that $\theta = n / \sum_{i=1}^{n} y_i$, or

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} Y_i} = \frac{1}{\overline{Y}}$$

3. (a) We see that

$$L(\theta) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} [(\theta{+}1) y_i^{\theta}] = (\theta{+}1)^n \prod_{i=1}^{n} y_i^{\theta} = (\theta{+}1)^n \left(\prod_{i=1}^{n} y_i\right)^{\theta} = g\left(\prod_{i=1}^{n} y_i, \theta\right) h(y_1, \ldots, y_n)$$

so that $(\prod_{i=1}^{n} Y_i)$ is a sufficient statistic for θ by the Factorization Theorem.

(b) We need to set

$$\begin{aligned} E(Y) &= \int y f_Y(y)\, dy = \int_0^1 y(\theta+1)y^{\theta}\, dy = \int_0^1 (\theta+1) y^{\theta+1}\, dy = \frac{(\theta+1)}{(\theta+2)} y^{\theta+2} \Big|_0^1 = \frac{(\theta+1)}{(\theta+2)} \\ &= \overline{Y} \end{aligned}$$

and solving for θ, we obtain

$$\hat{\theta} = \frac{2\overline{Y} - 1}{1 - \overline{Y}}$$

(c) (Bonus) First, we have from part (a) that

$$L(\theta) = (\theta + 1)^n \prod_{i=1}^{n} y_i^{\theta}$$

so that

$$\ln L(\theta) = n\ln(\theta + 1) + \ln \prod_{i=1}^{n} y_i^{\theta} = n\ln(\theta + 1) + \sum_{i=1}^{n} \ln y_i^{\theta} = n\ln(\theta + 1) + \theta \sum_{i=1}^{n} \ln y_i$$

and by letting

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{(\theta + 1)} + \sum_{i=1}^{n} \ln y_i = 0$$

and solving for θ, we obtain $\theta = -1 - n / \sum_{i=1}^{n} \ln y_i$. Hence,

$$\hat{\theta}_{MLE} = -1 - \frac{n}{\sum_{i=1}^{n} \ln Y_i}$$