Homework 3

Due Tuesday, February 6

1. Suppose that Y_1, Y_2, and Y_3 are iid exponential random variables with $\beta = 1$. Show that $U = Y_1 + Y_2 + Y_3$ follows Gamma$(3, 1)$ distribution, using the cdf method.

2. Repeat Problem 1, using the pdf (Jacobian) method.

3. If Y_1, Y_2, and Y_3 are iid exponential random variables with $\beta = 1$, derive the joint pdf of $U_1 = Y_1 + Y_2$, $U_2 = Y_1 + Y_3$, $U_3 = Y_2 + Y_3$.

4. Let Y be a binomial random variable with n trials and probability of success given by p. Show, using the mgf method, that $n - Y$ is a binomial random variable with n trials and probability of success given by $1 - p$.

5. Suppose that Y has a gamma distribution with $\alpha = n/2$ and $\beta = m$ for some positive integer n and m. Use the mgf method to show that $W = 2Y/m$ has a χ^2 distribution with n degrees of freedom, $\chi^2(n)$ (recall that $\chi^2(\nu) = \text{Gamma}(\nu/2, 2)$).

6. If mgf of Y is given by
$$M_Y(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$$
calculate $P(Y \leq 2)$.

7. Let Y_1 and Y_2 be independent Poisson random variables with means λ_1 and λ_2, respectively. Find the distribution of $U = Y_1 + Y_2$ using the mgf method.

8. If $Y_1 \sim$ Binomial(5,3/4) and $Y_2 \sim$ Poisson(2), and Y_1 and Y_2 are independent, find the joint mgf of Y_1 and Y_2, i.e., $M_{Y_1+Y_2}(t)$.

9. Let Y_1 and Y_2 be independent standard normal random variables. Find the distribution of $U = Y_1^2 + Y_2^2$ using the mgf method. (HINT: You may use the result of Example 6.11).

10. Suppose that $Y_1, \ldots, Y_n$ are iid random variables, each Y_i having a probability function
$$P(Y = y) = p^y(1-p)^{1-y}, \quad y = 0, 1$$
    (a) Show that $M_Y(t) = 1 - p + e^t p$.
    (b) Show that $Y_1 + \cdots + Y_n \sim \text{Binomial}(n, p)$.

11. Suppose that $Y_1, \ldots, Y_n$ are iid $N(\mu, \sigma^2)$ random variables. Compute the mgf of
$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$
and determine the distribution of $\overline{Y}$.

12. **(5880*)** Let $Y_1 \sim \text{Gamma}(\alpha_1, \beta)$, $Y_2 \sim \text{Gamma}(\alpha_2, \beta)$, $Y_3 \sim \text{Gamma}(\alpha_3, \beta)$ be independent. If $U = c_1 Y_1 + c_2 Y 2 + c_3 Y_3$, what condition must we impose on constants c_1, c_2, c_3 to ensure that U has a gamma distribution with the first parameter $\alpha_1 + \alpha_2 + \alpha_3$?

13. **(5880*)** Show that $Y \sim \text{Poisson}(\theta)$ belongs to an exponential family.

14. **(5880*)** Show that (a) $Y \sim N(\theta, 1)$ and (b) $Y \sim \text{Beta}(2, \theta)$ each belong to an exponential family.

15. **(5880*)** Show that $Y \sim \text{Beta}(\alpha, \beta)$ belongs to an exponential family.