Homework 4

Due Tuesday, February 13

1. Suppose that $X_1, X_2, ..., X_m$ and $Y_1, Y_2, ..., Y_n$ are independent random samples (independence between X_i and Y_j as well as within X_i and within Y_j), with the variables $X_i \sim N(\mu_X, \sigma_X^2)$ and the variables $Y_j \sim N(\mu_Y, \sigma_Y^2)$. The difference between the sample means, $\overline{X} - \overline{Y}$, is then itself normally distributed.
   (a) Find $E(\overline{X} - \overline{Y})$
   (b) Find $\text{Var}(\overline{X} - \overline{Y})$
2. If $Y_1, Y_2, ..., Y_n \sim$ iid $N(\mu, \sigma^2)$ and S^2 is the sample variance, let $U = (n-1)S^2/\sigma^2$. Recalling the distribution of U, compute $\frac{d}{dt} M_U(t)\big|_{t=0}$ to show that $E(S^2) = \sigma^2$.
3. If $Y_1, Y_2, ..., Y_n \sim$ iid $N(\mu, \sigma^2)$ and S^2 is the sample variance, find the moment generating function of S^2.
4. Show that
$$\sum_{i=1}^{n} [(Y_i - \overline{Y})(\overline{Y} - \mu)] = 0$$
(which proves that $\sum_{i=1}^{n} (Y_i - \mu)^2 = \sum_{i=1}^{n} (Y_i - \overline{Y})^2 + n(\overline{Y} - \mu)^2 + 0$).
5. Show that
$$(Y_1 - \overline{Y})^2 = \left[\sum_{i=2}^{n} (Y_i - \overline{Y}) \right]^2$$
6. Show that
$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (Y_i - Y_j)^2$$
(HINT: Work backwards, add and subtract $\overline{Y}$).
7. If Y has a t-distribution with n d.f., show that $Y^2 \sim F(1, n)$. (No need to derive the pdf; just recall the composition of Y and see what Y^2 looks like).
8. Let Y have the F distribution with m and n degrees of freedom (i.e., $Y \sim F(m, n)$). Find the distribution of $1/Y$. (Again, no need to derive the pdf).
9. If Y has an exponential distribution with mean θ, show that $U = 2Y/\theta$ has a χ^2 distribution with 2 df. Deduce that if X_1 and X_2 are independent exponential distributions with mean θ, then X_1/X_2 follows an F-distribution (with what degrees of freedom?).
10. **(5880*)** If $Y_1, Y_2, ..., Y_n \sim$ iid $N(\mu, \sigma^2)$, show that $\overline{Y}$ and S^2 are independent by following the steps below:

(a) Let $U_1 = \overline{Y}, U_2 = Y_2 - \overline{Y}, U_3 = Y_3 - \overline{Y}, \ldots, U_n = Y_n - \overline{Y}$. Find the inverse transformation Y_i, $i = 1, \ldots, n$, and show that the Jacobian $J = n$.

(b) Show that (hint: use Problem 5) the joint pdf of $(U_1, \ldots, U_n)$ is given by

$$f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) = n\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[\frac{-1}{2\sigma^2}\left(\left[\sum_{i=2}^{n} u_i\right]^2 + \sum_{i=2} u_i^2\right)\right] \exp\left[\frac{-n}{2\sigma^2}(u_1 - \mu)^2\right]$$

(c) Show that (hint: use part (b))

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \frac{1}{n-1}\left(\left[\sum_{i=2}^{n} U_i\right]^2 + \sum_{i=2}^{n} U_i^2\right)$$

and deduce that $\overline{Y}$ and S^2 are independent.

11. **(5880*)** We derive the pdf of a t-distribution by following the steps below:

(a) Let $Y_1 \sim N(0, 1)$ and $Y_2 \sim \chi^2(r)$ be independent, and let $U_1 = Y_1/\sqrt{Y_2/r}$. By letting $U_2 = Y_2$, show that the Jacobian is $J = \sqrt{u_2/r}$.

(b) Show that the joint pdf of (U_1, U_2) is given by

$$f_{U_1,U_2}(u_1, u_2) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r+1}{2}}\sqrt{\pi r}} u_2^{\frac{r+1}{2}-1} e^{-\frac{[1+(u_1^2/r)]u_2}{2}}, \quad -\infty < u_1 < \infty, \quad u_2 > 0$$

(c) Show that the marginal density of U_1 is given by

$$f_{U_1}(u_1) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})\sqrt{\pi r}} \frac{1}{[1 + (u_1^2/r)]^{\frac{r+1}{2}}}, \quad -\infty < u_1 < \infty$$

which is the pdf of the t-distribution with r d.f. (as a hint, recall the form of the gamma pdf and use it to integrate over u_2) .

12. **(5880*)** We derive the pdf of an F-distribution by following the steps below:

(a) Let $Y_1 \sim \chi^2(m)$ and $Y_2 \sim \chi^2(n)$ be independent, and let $U_1 = Y_1/Y_2$ and $U_2 = Y_2$. Show that the Jacobian is $J = u_2$.

(b) Show that the joint pdf of (U, V) is given by

$$f_{U_1,U_2}(u_1, u_2) = \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u^{m/2-1} u_2^{(m+n)/2-1} e^{-(1+u_1)u_2/2}, \quad u_1 > 0, u_2 > 0$$

(c) Show that the marginal density of U_1 is given by (again, recall the form of the gamma pdf and use it to integrate over u_2)

$$f_{U_1}(u_1) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{u_1^{m/2-1}}{(1+u)^{(m+n)/2}}, \quad u_1 > 0$$

and by letting $W = (Y_1/m)/(Y_2/n)$, deduce that $f_W(w)$ has the pdf of $F(m, n)$

$$f_W(w) \quad = \quad \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} \frac{w^{m/2-1}}{\left(1 + \left(\frac{m}{n}\right)w\right)^{(m+n)/2}}, \quad w > 0.$$