Homework 6
Due Tuesday, March 12

1. Let Y be a random variable and let $U = \ln Y$, with $E(U) = 0$. Show that $E(Y) \geq 1$.
2. Show that $E(S) \leq \sigma$.
3. (a) Show that $|E(Y)| \leq E(|Y|)$. (HINT: You may assume that the absolute value function is convex).
   (b) Use Cauchy-Schwarz inequality
   $$E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}$$
   and part (a) to show that
   $$|\text{Cov}(Y_1, Y_2)| \leq \sqrt{\text{Var}(Y_1)}\sqrt{\text{Var}(Y_2)}$$
   (c) Use part (b) to conclude that the correlation ρ between two random variables Y_1 and Y_2 is always between -1 and 1, i.e., $-1 \leq \rho \leq 1$.
4. Assume that $\text{Var}(S^2) \to 0$ as $n \to \infty$. Show that for any $\epsilon > 0$, $P(|S^2 - \sigma^2| \geq \epsilon) \to 0$ (and hence $S_n^2 \to \sigma^2$ in probability) as $n \to \infty$.
5. Let $X_1, \ldots, X_n \sim$ iid Binomial$(1, p)$, and let
   $$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$
   (a) Show that $E(X_i^k) = p$ for any integer k.
   (b) Show that $E(Y_n) = p$.
   (c) Show that $Y_n \to p$ in probability.
6. Suppose that $Y \sim$ Poisson(λ).
   (a) Let $U = (Y - \lambda)/\sqrt{\lambda}$. Show that $M_U(t) = \exp(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda)$.
   (b) Show that $M_U(t) \to \exp(t^2/2)$ as $\lambda \to \infty$ (HINT: Use $e^a = 1 + a + a^2/2! + a^3/3! + \cdots$ to expand $e^{t/\sqrt{\lambda}}$).
7. Let $Y_1, Y_2, \ldots$ be a sequence of random variables with probability function
   $$P(Y_n = 0) = 1 - \frac{1}{n} \quad \text{and} \quad P(Y_n = n^2) = \frac{1}{n}$$
   Show that $\lim_{n\to\infty} E(Y_n) = \infty$.

8. **(5880*)** Let $X_1, \ldots, X_n \sim$ iid Uniform$(0, \theta)$. Let $Y_n = n(\theta - X_{(n)})$, where $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Find the cdf of Y_n, and show that $Y_n \to Y$ in distribution, where $Y \sim$ Exponential(θ).

9. **(5880*)** Prove the central limit theorem (CLT). Specifically, if $Y_1, \ldots, Y_n$ are iid with $E(Y_i) = \mu$ and $\text{Var}(Y_i) = \sigma^2 < \infty$, then let

$$U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n Y_i - n\mu}{\sqrt{n}\sigma}$$

and show that

$$U_n \to Z \sim N(0, 1)$$

in distribution (or equivalently, $M_{U_n}(t) \to M_Z(t)$) as $n \to \infty$, by following the steps below.

(a) We may assume that $\mu = E(Y_i) = 0$, so that $U_n = \frac{\sum_{i=1}^n Y_i}{\sqrt{n}\sigma}$. Show that

$$M_{U_n}(t) = \left(E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_1\right)\right]\right)^n = \left(M_{Y_1}\left(\frac{t}{\sqrt{n}\sigma}\right)\right)^n$$

(b) Using the fact that $e^a = 1 + a + a^2/2! + \cdots$, show that

$$M_{Y_1}\left(\frac{t}{\sqrt{n}\sigma}\right) = E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_1\right)\right] = 1 + \frac{t^2}{2n} + \cdots$$

(c) Assume that the "$\cdots$" terms are small, so that $M_{Y_1}\left(\frac{t}{\sqrt{n}\sigma}\right) = 1 + \frac{t^2}{2n}$. Show that

$$M_{U_n}(t) \to e^{t^2/2} = M_Z(t) \quad \text{(where } M_Z(t) \text{ is the mgf of } Z \sim N(0,1))$$

as $n \to \infty$, and thus proving that $U_n \to Z$ in distribution.

10. **(5880*)** Suppose that $Y_1, \ldots, Y_n$ are iid random variables with $E(Y_i) = \mu \neq 0$ and $\text{Var}(Y_i) = \sigma^2 < \infty$. Find the limiting distribution of $\sqrt{n}(1/\overline{Y} - 1/\mu)$, i.e., find "?" if $\sqrt{n}(1/\overline{Y} - 1/\mu) \xrightarrow{d} ?$

11. **(5880*)** If $Y_1, \ldots, Y_n \sim$ iid Exponential(θ) with $\theta > 0$, find the limiting distribution of $\sqrt{n}(\overline{Y}^2 - \theta^2)$.

12. **(5880*)** Suppose that $X_1, \ldots, X_n$ are iid Bernoulli(p), and let $Y_n = \overline{X}$.

(a) Show that $\sqrt{n}(Y_n - p) \xrightarrow{d} N[0, p(1-p)]$.

(b) If $p \neq 1/2$, find the limiting distribution of $\sqrt{n}[Y_n(1 - Y_n) - p(1-p)]$.

(c) If $p = 1/2$, find the limiting distribution of $n[Y_n(1 - Y_n) - 1/4]$.