Homework 7

Due Tuesday, March 26

1. Show that
$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2$$
(HINT: Subtract and add $E(\hat{\theta})$ inside $E[(\hat{\theta} - \theta)^2]$ to start).

2. Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function
$$f_Y(y) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0$$
Consider the following five estimators of θ
$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = \min\{Y_1, Y_2, Y_3\}, \quad \hat{\theta}_5 = \overline{Y}$$
   (a) Show that $E(\hat{\theta}_4) = \theta/3$, and hence $\hat{\theta}_4$ is biased.
   (b) Among the **unbiased** estimators, which has the smallest variance?

3. Let $Y_1, Y_2, \ldots, Y_n$ be random sample whose density is given by
$$f_Y(y) = \frac{\alpha y^{\alpha - 1}}{\theta^\alpha}, \quad 0 \le y \le \theta$$
where $\alpha > 0$ is a known fixed value, but θ is unknown (fixed). Consider the estimator $\hat{\theta} = \max\{Y_1, Y_2, \ldots, Y_n\}$.
   (a) Show that the cdf of Y is $F_Y(y) = (y/\theta)^\alpha$, $0 \le y \le \theta$.
   (b) Compute $E(\hat{\theta})$ to show that $\hat{\theta}$ is a *biased* estimator for θ.
   (c) Find a multiple of $\hat{\theta}$ that is an unbiased estimator of θ.

4. If Y has a binomial distribution with parameters n and p, then $\hat{p}_1 = Y/n$ is an unbiased estimator of p. Another estimator of p is $\hat{p}_2 = (Y + 1)/(n + 2)$.
   (a) Derive the bias of $\hat{p}_2$.
   (b) Derive $\text{MSE}(\hat{p}_1)$ and $\text{MSE}(\hat{p}_2)$.

5. Let $Y_1, Y_2, \ldots, Y_n \sim$ iid Uniform$(0, \theta)$. Consider $\hat{\theta} = Y_{(1)} = \min\{Y_1, Y_2, \ldots, Y_n\}$. Show that
$$E(\hat{\theta}) = \frac{\theta}{n+1} \quad \text{and} \quad \text{Var}(\hat{\theta}) = \frac{n\theta^2}{(n+1)^2(n+2)}$$
Find an unbiased estimator of θ based on this result.

6. Suppose that $Y_1, Y_2, \ldots, Y_n$ are iid with density function

$$f_Y(y) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0$$

   (a) Show that $\hat{\theta}_1 = nY_{(1)}$ is an unbiased estimator for θ, and find MSE($\hat{\theta}_1$). (HINT: You may recognize the pdf of $Y_{(1)}$ and use its properties).

   (b) Show that $\hat{\theta}_2 = \overline{Y}$ is an unbiased estimators of θ.

   (c) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

7. Let $Y_1, Y_2, \ldots, Y_n \sim$ iid Uniform$(0, \theta)$. We have seen that

$$\hat{\theta}_1 = (n+1)Y_{(1)} \quad \text{and} \quad \hat{\theta}_2 = \left(\frac{n+1}{n}\right) Y_{(n)}$$

   are unbiased estimators for θ.

   (a) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ (see Problem 5 and Example 9.1 on page 446 of textbook; you may state and use the results from there).

   (b) Show that $\hat{\theta}_2$ is a consistent estimator for θ. Deduce that $Y_{(n)}$ is a consistent estimator of θ and also asymptotically unbiased estimator for θ.

8. If Y has a binomial distribution with parameters n and p, consider $\hat{p} = (Y+1)/(n+2)$ as an estimator of p. Is $\hat{p}$ a consistent estimator of p? Is $\hat{p}$ an asymptotically unbiased estimator for p?

9. Suppose that $X_1, X_2, ..., X_m$ and $Y_1, Y_2, ..., Y_n$ are independent random samples from population with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively (variances are finite). Show that $\overline{X} - \overline{Y}$ is a consistent estimator of $\mu_1 - \mu_2$.

10. Let $Y_1, Y_2, \ldots, Y_n$ be iid Gamma(α, β). Show that $\overline{Y}$ is a consistent estimator of $\alpha\beta$.

11. **(5880*)** Let $Y_1, Y_2, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$. Consider the following estimators of σ^2

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

   Find MSE(S^2) and MSE($\hat{\sigma}_n^2$). (HINT: Since $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, use the property of the χ^2 distribution to find the variance of S^2).

12. **(5880*)** Let $Y_1, Y_2, \ldots, Y_n$ be iid random variables, each with pdf

$$f_Y(y) = \left(\frac{2y}{\theta}\right) e^{-y^2/\theta}, \quad y > 0$$

   where $\theta > 0$.

   (a) Find the distribution of Y^2.

   (b) Show that $W_n = \frac{1}{n} \sum_{i=1}^{n} Y_i^2$ is a consistent estimator for θ.

(The following two problems will require the independent reading of Cramér-Rao Inequality notes, on course webpage)

13. **(5880*)** Let $Y_1, Y_2, \ldots, Y_n$ be iid random variables, each with pdf

$$f_Y(y) = \theta y^{(\theta-1)}, \quad 0 < y < 1, \quad \theta > 0$$

   (a) Show that Y belongs to an exponential family, and identify its components.
   (b) Compute $E(Y)$.
   (c) Compute the Fisher Information.
   (d) Compute the Cramér-Rao Lower Bound for $\text{Var}(\overline{Y})$.

14. **(5880*)** Suppose that $Y_1, Y_2, \ldots, Y_n$ are iid Bernoulli(p) random variables. Show that $\hat{p} = \overline{Y}$ is an efficient estimator of p.