2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc \frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

4. (a) $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$ since $\sec \frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

6. (a) Let $\theta = \arctan 2$, so $\tan\theta = 2 \quad \Rightarrow \quad \sec^2\theta = 1 + \tan^2\theta = 1 + 4 = 5 \quad \Rightarrow \quad \sec\theta = \sqrt{5} \quad \Rightarrow$ $\sec(\arctan\, 2) = \sec\theta = \sqrt{5}$.

(b) Let $\theta = \sin^{-1}\left(\frac{5}{13}\right)$. Then $\sin\theta = \frac{5}{13}$, so $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = \frac{119}{169}$.

14. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y \, \tan y \left(\frac{dy}{dx}\right) = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \, \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2 - 1}}$. Note that $\tan^2 y = \sec^2 y - 1 \quad \Rightarrow$ $\tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

24. $y = x\cos^{-1} x - \sqrt{1 - x^2} \quad \Rightarrow \quad y' = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}} + \dfrac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

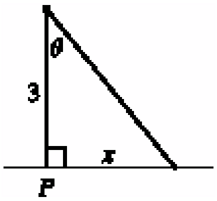
30. $f(x) = \arcsin(e^x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{1 - (e^x)^2}} \cdot e^x = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$.

$\text{Domain}(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

$\text{Domain}(f') = \left\{x \mid 1 - e^{2x} > 0\right\} = \left\{x \mid e^{2x} < 1\right\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

38. Let $t = \ln x$. As $x \to 0^+$, $t \to -\infty$. $\lim\limits_{x \to 0^+} \tan^{-1}(\ln x) = \lim\limits_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2}$ by (8).

40.

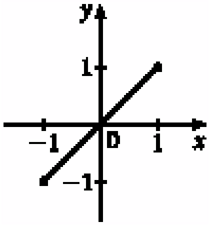


$\dfrac{d\theta}{dt} = 4 \text{ rev/min} = 8\pi \cdot 60 \text{ rad/h}$. From the diagram, we see that $\tan\theta = \dfrac{x}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{x}{3}\right)$.

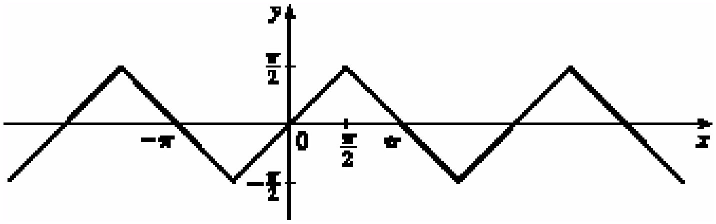
Thus, $8\pi \cdot 60 = \dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/3}{1 + (x/3)^2}\dfrac{dx}{dt}$. So $\dfrac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left[1 + \left(\dfrac{x}{3}\right)^2\right]$ km/h, and

at $x = 1$, $\dfrac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left[1 + \frac{1}{9}\right]$ km/h $= 1600\pi$ km/h.

42. (a) $f(x) = \sin(\sin^{-1} x)$



(b) $g(x) = \sin^{-1}(\sin x)$



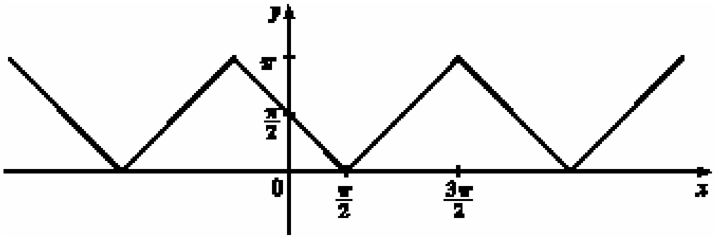
(c) $g'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x) = \dfrac{1}{\sqrt{1-\sin^2 x}}\cos x = \dfrac{\cos x}{\sqrt{\cos^2 x}} = \dfrac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so

$$h'(x) = -\frac{\cos x}{\sqrt{1-\sin^2 x}} = -\frac{\cos x}{|\cos x|}.$$

Notice that $h(x) = \frac{\pi}{2} - g(x)$ because

$\sin^{-1} t + \cos^{-1} t = \frac{\pi}{2}$ for all t.



12. $$\begin{aligned}\cosh x\,\cosh y + \sinh x\,\sinh y &= \left[\tfrac{1}{2}\left(e^{x}+e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^{y}+e^{-y}\right)\right] + \left[\tfrac{1}{2}\left(e^{x}-e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^{y}-e^{-y}\right)\right] \\ &= \tfrac{1}{4}\left[\left(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}\right)+\left(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}\right)\right] \\ &= \tfrac{1}{4}\left(2e^{x+y}+2e^{-x-y}\right) = \tfrac{1}{2}[e^{x+y}+e^{-(x+y)}] = \cosh(x+y)\end{aligned}$$

14. $$\begin{aligned}\frac{1+\tanh x}{1-\tanh x} &= \frac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \frac{\cosh x+\sinh x}{\cosh x-\sinh x} = \frac{\frac{1}{2}\left(e^{x}+e^{-x}\right)+\frac{1}{2}\left(e^{x}-e^{-x}\right)}{\frac{1}{2}\left(e^{x}+e^{-x}\right)-\frac{1}{2}\left(e^{x}-e^{-x}\right)} \\ &= \frac{e^{x}+e^{-x}+e^{x}-e^{-x}}{e^{x}+e^{-x}-e^{x}+e^{-x}} = \frac{2e^{x}}{2e^{-x}} = e^{2x}\end{aligned}$$

Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x+\sinh x}{\cosh x-\sinh x} = \dfrac{e^{x}}{e^{-x}} = e^{2x}$

16. $\sinh x = \frac{3}{4} \;\Rightarrow\; \operatorname{csch} x = 1/\sinh x = \frac{4}{3}$. $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16}+1 = \frac{25}{16} \;\Rightarrow\; \cosh x = \frac{5}{4}$ (since $\cosh x > 0$).

$\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

22. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2-1}$. So, by Exercise 9,

$e^{y} = \cosh y + \sinh y = x + \sqrt{x^2-1} \;\Rightarrow\; y = \ln\left(x+\sqrt{x^2-1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^{y}+e^{-y}\right)$ and solve a quadratic, as in Example 3.

32. $f(t) = \ln(\sinh t) \;\Rightarrow\; f'(t) = \dfrac{1}{\sinh t}\cosh t = \coth t$

43. (a) $y = 20\cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha = \text{slope of the line} = \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343 \text{ rad} \approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

44. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{T}{\rho g}\sinh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \sinh\frac{\rho g x}{T} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}.$$

We evaluate the two sides separately: $\text{LHS} = \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$,

$\text{RHS} = \dfrac{\rho g}{T}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1 + \sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

46. $$\lim_{x\to\infty}\frac{\sinh x}{e^x} = \lim_{x\to\infty}\frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty}\frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$$

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$

2. $$\lim_{x\to 1}\frac{x^a - 1}{x^b - 1} \overset{\text{H}}{=} \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

8. $$\lim_{\theta\to\pi/2}\frac{1 - \sin\theta}{\csc\theta} = \frac{0}{1} = 0.$$ L'Hospital's Rule does not apply.

14. This limit has the form $\frac{\infty}{\infty}$. $$\lim_{x\to\infty}\frac{e^x}{x^3} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{3x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{6x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

16. $$\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-m\sin mx + n\sin nx}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-m^2\cos mx + n^2\cos nx}{2} = \tfrac{1}{2}\left(n^2 - m^2\right)$$

30. $$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1}\frac{x - 1 - \ln x}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1}\frac{1 - 1/x}{(x-1)(1/x) + \ln x}\cdot\frac{x}{x} \\ &= \lim_{x\to 1}\frac{x-1}{x - 1 + x\ln x} \overset{\text{H}}{=} \lim_{x\to 1}\frac{1}{1 + 1 + \ln x} = \frac{1}{2+0} = \frac{1}{2}\end{aligned}$$

34. $y = \left(1+\frac{a}{x}\right)^{bx} \quad\Rightarrow\quad \ln y = bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} \overset{\mathrm{H}}{=} \lim_{x\to\infty} \frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1+a/x} = ab \quad\Rightarrow$$

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

40. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\mathrm{H}}{=} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p > 0$.

44. (a) $\displaystyle\lim_{R\to r^+} v = \lim_{R\to r^+} \left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -cr^2 \lim_{R\to r^+} \left[\left(\frac{1}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -cr^2 \cdot \frac{1}{r^2} \cdot \ln 1 = -c \cdot 0 = 0$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\displaystyle\lim_{r\to 0^+} v = \lim_{r\to 0^+} \left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -\frac{c}{R^2} \lim_{r\to 0^+} \left[r^2 \ln\left(\frac{r}{R}\right)\right]$ [form is $0\cdot\infty$]

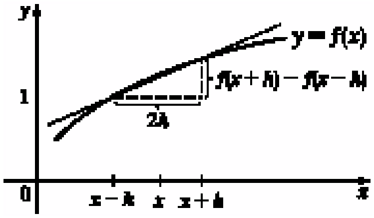
$$= -\frac{c}{R^2} \lim_{r\to 0^+} \frac{\ln\left(\dfrac{r}{R}\right)}{\dfrac{1}{r^2}} \qquad \text{[form is } \infty/\infty\text{]}$$

$$\overset{\mathrm{H}}{=} -\frac{c}{R^2} \lim_{r\to 0^+} \frac{\dfrac{R}{r}\cdot\dfrac{1}{R}}{\dfrac{-2}{r^3}} = -\frac{c}{R^2} \lim_{r\to 0^+} \left(-\frac{r^2}{2}\right) = 0$$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

49. Since $\displaystyle\lim_{h\to 0}[f(x+h) - f(x-h)] = f(x) - f(x) = 0$ (f is differentiable and hence continuous) and $\displaystyle\lim_{h\to 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \overset{\mathrm{H}}{=} \lim_{h\to 0} \frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \frac{f'(x) + f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$



$\dfrac{f(x+h) - f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h \to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.

50. Since $\lim_{h\to 0}[f(x+h)-2f(x)+f(x-h)] = f(x)-2f(x)+f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{\mathrm{H}}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 49 to $f'(x)$.