Integration with computers

Indefinite integration

Polynomials and rational functions

For polynomials, computer algebra systems like *Mathematica* do a fine job of antidifferentiating.

```
Integrate[x^2, x]
```

$$\frac{x^3}{3}$$

Note, though, that *Mathematica* gives only a single antiderivative; it omits the constant of integration.

But as soon as we move beyond polynomials to rational functions, there can be problems.

```
Integrate[1 / x, x]
```

```
Log[x]
```

Note that the computer's answer in this case is only a valid antiderivative for positive values of *x*. (FYI, *Mathematica* uses `Log[x]` to mean natural log.)

Let's explore this problem further.

The (natural) log, absolute value, and Heaviside functions

```
D[Log[x], x]
```

$$\frac{1}{x}$$

(Well, at least *Mathematica* is consistent...)

```
D[Log[Abs[x]], x]
```

$$\frac{\mathrm{Abs}'(x)}{|x|}$$

This is true, but not very helpful.

What about antidifferentiating |*x*|?

```
Integrate[Abs[x], x]
```

$$\int |x|\, dx$$

This is *Mathematica*'s way of saying "I don't know how to symbolically antidifferentiate that."

In fact, we know that (1/2) *x* |*x*| is an antiderivative of |*x*|; can we get *Mathematica* to at least recognize this?

```
D[(1 / 2) x Abs[x], x]
```

$$\frac{\text{Abs}[x]}{2} + \frac{1}{2}\, x\, \text{Abs}'[x]$$

```
Simplify[%]
```

$$\frac{1}{2}\left(\text{Abs}[x] + x\, \text{Abs}'[x]\right)$$

Guess not!

Integration by partial fractions

Let's try that integration-by-partial fractions example.

```
Integrate[(3 x^3 - 3 x^2 + x + 1) / (x^4 - 2 x^3 + 2 x^2 - 2 x + 1), x]
```

$$\frac{1}{1-x} + \text{ArcTan}[x] + \text{Log}[-1+x] + \text{Log}\left[1+x^2\right]$$

```
D[1 / (1 - x) + ArcTan[x] + Log[-1 + x] + Log[1 + x^2], x]
```

$$\frac{1}{(1-x)^2} + \frac{1}{-1+x} + \frac{1}{1+x^2} + \frac{2x}{1+x^2}$$

We need to tell *Mathematica* to combine terms:

```
Together[%]
```

$$\frac{1 + x - 3x^2 + 3x^3}{(-1+x)^2\left(1+x^2\right)}$$

Okay, but we had to explicitly prompt *Mathematica* to write the answer in a friendly form.

Polynomials revisited

I said before that *Mathematica* does a fine job of antidifferentiating, but in a sense this isn't true.

```
D[(1 / 18) (x^2 + 5)^9, x]
```

$$x\left(5+x^2\right)^8$$

```
Integrate[%, x]
```

$$\frac{390625\,x^2}{2}+156250\,x^4+\frac{218750\,x^6}{3}+21875\,x^8+4375\,x^{10}+\frac{1750\,x^{12}}{3}+50\,x^{14}+\frac{5\,x^{16}}{2}+\frac{x^{18}}{18}$$

```
Simplify[% - (1 / 18) (x^2 + 5)^9]
```

$$-\frac{1953125}{18}$$

Here *Mathematica* has not chosen a good (i.e., nicely factoring) antiderivative.

Human-computer collaboration on an indefinite integral

Here's an integral that you can do but *Mathematica* can't.

```
Integrate[(1 + Log[x]) Sqrt[1 - (x * Log[x])^2], x]
```

$$\int (1+\text{Log}[x])\sqrt{1-x^2\,\text{Log}[x]^2}\,dx$$

Here *Mathematica* fails to guess the correct substitution.

We can help *Mathematica* out of a jam by giving it the correct substitution:

Put $u = x \ln x$, $du = (1 + \ln x)\, dx$.

```
Integrate[Sqrt[1 - u^2], u]
```

$$\frac{1}{2}\left(u\sqrt{1-u^2}+\text{ArcSin}[u]\right)$$

```
% /. u → x Log[x]
```

$$\frac{1}{2}\left(\text{ArcSin}[x\,\text{Log}[x]]+x\,\text{Log}[x]\sqrt{1-x^2\,\text{Log}[x]^2}\right)$$

```
D[%, x]
```

$$\frac{1}{2}\left(\frac{1+\text{Log}[x]}{\sqrt{1-x^2\,\text{Log}[x]^2}}+\frac{x\,\text{Log}[x]\left(-2\,x\,\text{Log}[x]-2\,x\,\text{Log}[x]^2\right)}{2\sqrt{1-x^2\,\text{Log}[x]^2}}+\sqrt{1-x^2\,\text{Log}[x]^2}+\text{Log}[x]\sqrt{1-x^2\,\text{Log}[x]^2}\right)$$

```
Simplify[%]
```

$$(1+\text{Log}[x])\sqrt{1-x^2\,\text{Log}[x]^2}$$

This enhances our confidence in the answer

$$\frac{1}{2}\left(\text{ArcSin}[x\,\text{Log}[x]]+x\,\text{Log}[x]\sqrt{1-x^2\,\text{Log}[x]^2}\right)$$

This is a great example of human and computer working together to achieve a task that would be hard for a human alone or a computer alone.

Definite integration

The (natural) log, absolute value, and Heaviside functions

Definite integrals are often handled by different routines in a computer algebra system than indefinite integrals. For instance, *Mathematica* does not always evaluate definite integrals by using the Evaluation Theorem. We can see this with an example.

```
Integrate[1 / x, {x, 1, 2}]
```

```
Log[2]
```

```
Integrate[1 / x, {x, -2, -1}]
```

```
-Log[2]
```

Mathematica gets the right answer for the definite integral, even though it didn't for the indefinite integral.

(What about integrating from -1 to 1?

```
Integrate[1 / x, {x, -1, 1}]
```

Integrate::idiv : Integral of $\frac{1}{x}$ does not converge on $\{-1, 1\}$. ≫

$$\int_{-1}^{1} \frac{1}{x}\, dx$$

Mathematica correctly reports that the integral doesn't converge.)

What about integrating the absolute value function?

```
Integrate[Abs[x], {x, -1, 1}]
```

1

Ditto.

Can we use "definite" integration with indefinite limits of integration to get an antiderivative by the back door (i.e. via the FTC)?

```
Integrate[Abs[t], {t, 1, x}]
```

$$-\Big((x-1)\Big((\mathrm{Re}(x)-1)\Big(\sqrt{\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2}-\mathrm{Re}(x)\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}\Big)-\mathrm{Im}(x)^2\Big(\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}-\log\Big(\sqrt{\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2}+\mathrm{Re}(x)-1\Big)+\log\Big(\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}+\mathrm{Im}(x)^2+\mathrm{Re}(x)^2-\mathrm{Re}(x)\Big)\Big)\Big)\Big)\Big/\Big(2\left(\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2\right)^{3/2}\Big)$$

Ugh!

```
Simplify[%]
```

$$-\Big((x-1)\Big((\mathrm{Re}(x)-1)\Big(\sqrt{\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2}-\mathrm{Re}(x)\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}\Big)-\mathrm{Im}(x)^2\Big(\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}-\log\Big(\sqrt{\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2}+\mathrm{Re}(x)-1\Big)+\log\Big(\sqrt{\left(\mathrm{Im}(x)^2\left(2\,\mathrm{Re}(x)^2-2\,\mathrm{Re}(x)+1\right)+\mathrm{Im}(x)^4+(\mathrm{Re}(x)-1)^2\,\mathrm{Re}(x)^2\right)}+\mathrm{Im}(x)^2+\mathrm{Re}(x)^2-\mathrm{Re}(x)\Big)\Big)\Big)\Big)\Big/\Big(2\left(\mathrm{Im}(x)^2+(\mathrm{Re}(x)-1)^2\right)^{3/2}\Big)$$

To see if this is correct, try to evaluate it at $x = 1$ and see if we get (1/2) 1 |1| = 1/2:

```
% /. x → 1
```

Power::infy : Infinite expression $\frac{1}{0^{3/2}}$ encountered. ≫

∞::indet : Indeterminate expression $-\infty + \infty$ encountered. ≫

Indeterminate

It appears that *Mathematica* has created an expression involving the variable *x* that it can't evaluate for any particular value of *x*!

Human-computer collaboration on a definite integral

How does *Mathematica* do with that $u = x \ln x$ example?

```
Integrate[(1 + Log[x]) Sqrt[1 - (x * Log[x])^2], {x, 0.9, 1.1}]
```

$$\int_{0.9}^{1.1} (\log(x) + 1) \sqrt{1 - x^2 \log^2(x)}\, dx$$

It takes *Mathematica* half a minute or so to find this (useless) answer.

When we ask *Mathematica* to do this integral numerically, we get an answer quickly.

```
N[%]
```

0.199331

And this seems to be right, if we apply the Evaluation Theorem using the antiderivative we found before:

```
((1 / 2) (ArcSin[x Log[x]] + x Log[x] Sqrt[1 - x^2 Log[x]^2]) /. x → 1.1) -
 ((1 / 2) (ArcSin[x Log[x]] + x Log[x] Sqrt[1 - x^2 Log[x]^2]) /. x → 0.9)
```

0.199331

For some ideas about how did *Mathematica* might have done the calculation of the definite integral (without knowledge of the antiderivative), see section 6.5 of Stewart, which discusses numerical methods of integration.