Integration with computers

Indefinite integration

Polynomials and rational functions

For polynomials, computer algebra systems like Mathematica do a fine job of antidifferentiating.

\[
\text{Integrate}[x^2, x]
\]

\[
\frac{x^3}{3}
\]

Note, though, that Mathematica gives only a single antiderivative; it omits the constant of integration.

But as soon as we move beyond polynomials to rational functions, there can be problems.

\[
\text{Integrate}[1/x, x]
\]

\[
\log|x|
\]

Note that the computer's answer in this case is only a valid antiderivative for positive values of \(x\). (FYI, Mathematica uses \(\log[x]\) to mean natural log.)

Let's explore this problem further.

The (natural) log, absolute value, and Heaviside functions

\[
\text{D}[\log[x], x]
\]

\[
\frac{1}{x}
\]

(Well, at least Mathematica is consistent...)

\[
\text{D}[\text{Abs}[x], x]
\]

\[
\frac{\text{Abs}'(x)}{|x|}
\]

This is true, but not very helpful.
What about antidifferentiating $|x|$?

\[ \int |x| \, dx \]

This is *Mathematica*'s way of saying "I don't know how to symbolically antidifferentiate that."

In fact, we know that $(1/2) \ x \ |x|$ is an antiderivative of $|x|$; can we get *Mathematica* to at least recognize this?

\[ D[(1/2) \ x \ Abs[x], \ x] \]

\[ \frac{Abs[x]}{2} + \frac{-x \ Abs'[x]}{2} \]

\[ \text{Simplify[]} \]

\[ \frac{1}{2} (Abs[x] + x \ Abs'[x]) \]

**Guess not!**

Integration by partial fractions

Let's try that integration-by-partial fractions example.

\[ \int \frac{(3 x^3 - 3 x^2 + x + 1)}{(x^4 - 2 x^3 + 2 x^2 - 2 x + 1)} \, dx \]

\[ \frac{1}{1-x} + \text{ArcTan}[x] + \text{Log}[-1+x] + \text{Log}[1+x^2] \]

\[ \frac{1}{(1-x)^2} + \frac{1}{-1+x} + \frac{1}{1+x^2} + \frac{2 x}{1+x^2} \]

**We need to tell *Mathematica* to combine terms:**

\[ \text{Together}[] \]

\[ \frac{1 + x - 3 x^2 + 3 x^3}{(-1 + x)^2 (1 + x^2)} \]

Okay, but we had to explicitly prompt *Mathematica* to write the answer in a friendly form.
Polynomials revisited

I said before that *Mathematica* does a fine job of antidifferentiating, but in a sense this isn't true.

\[
D[(1/18) (x^2 + 5)^9, x] = x (5 + x^2)^8
\]

\[
\text{Integrate}[%, x] = \frac{390625 x^2}{2} + \frac{156250 x^4}{3} + \frac{21875 x^6}{3} + \frac{43750 x^8}{5} + \frac{1750 x^{12}}{2} + \frac{5 x^{16}}{18} + x^{18}
\]

\[
\text{Simplify}[% - (1/18) (x^2 + 5)^9] = \frac{1953125}{18}
\]

Here *Mathematica* has not chosen a good (i.e., nicely factoring) antiderivative.

Human-computer collaboration on an indefinite integral

Here's an integral that you can do but *Mathematica* can't.

\[
\int (1 + \log[x]) \sqrt{1 - x^2 \log[x]^2} \, dx
\]

Here *Mathematica* fails to guess the correct substitution.

We can help *Mathematica* out of a jam by giving it the correct substitution:

Put \( u = x \ln x \), \( du = (1 + \ln x) \, dx \).

\[
\text{Integrate}[\sqrt{1 - u^2}, u] = \frac{1}{2} \left( u \sqrt{1 - u^2} + \arcsin[u] \right)
\]

\[
% / . \ u \to x \log[x]
\]

\[
\frac{1}{2} \left( \arcsin[x \log[x]] + x \log[x] \sqrt{1 - x^2 \log[x]^2} \right)
\]
\[
\frac{1}{2} \left( \frac{1 + \log|x|}{\sqrt{1 - x^2 \log|x|^2}} + \frac{x \log|x| \left(-2 \times \log|x| - 2 \times \log|x|^2\right)}{2 \sqrt{1 - x^2 \log|x|^2}} + \sqrt{1 - x^2 \log|x|^2} + \log|x| \sqrt{1 - x^2 \log|x|^2} \right)
\]

Simplify:

\[
(1 + \log|x|) \sqrt{1 - x^2 \log|x|^2}
\]

This enhances our confidence in the answer:

\[
\frac{1}{2} \left( \text{ArcSin}[x \log|x|] + x \log|x| \sqrt{1 - x^2 \log|x|^2} \right)
\]

This is a great example of human and computer working together to achieve a task that would be hard for a human alone or a computer alone.

**Definite integration**

The (natural) log, absolute value, and Heaviside functions.

Definite integrals are often handled by different routines in a computer algebra system than indefinite integrals. For instance, *Mathematica* does not always evaluate definite integrals by using the Evaluation Theorem. We can see this with an example.

\[
\text{Integrate}[1/x, \{x, 1, 2\}]
\]

\[
\text{Log}[2]
\]

\[
\text{Integrate}[1/x, \{x, -2, -1\}]
\]

\[-\text{Log}[2]\]

*Mathematica* gets the right answer for the definite integral, even though it didn't for the indefinite integral.

(What about integrating from -1 to 1?)
\textbf{Integrate} \[1/x, \{x, -1, 1\}\]

Integrate::idiv: Integral of \(\frac{1}{x}\) does not converge on \((-1, 1)\). \(\Rightarrow\)

\[\int_{-1}^{1} \frac{1}{x} \, dx\]

\textit{Mathematica} correctly reports that the integral doesn't converge.\(\)

What about integrating the absolute value function?\(\)

\textbf{Integrate} [Abs\[x\], \{x, -1, 1\}\]

1

Ditto.\(\)

Can we use "definite" integration with indefinite limits of integration to get an antiderivative by the back door (i.e. via the FTC)?\(\)

\textbf{Integrate} [Abs\[t\], \{t, 1, x\}\]

\[-(x - 1) \left( (\text{Re}(x) - 1) \left( \sqrt{\text{Im}(x)^2 + (\text{Re}(x) - 1)^2} - \text{Re}(x) \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) \right) \right. - \left. \right. \left( \text{Im}(x)^2 \left( \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) - \log \left( \sqrt{\text{Im}(x)^2 + (\text{Re}(x) - 1)^2} + \text{Re}(x) - 1 \right) + \log \left( \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) + \text{Im}(x)^2 + \text{Re}(x)^2 - \text{Re}(x) \right) \right) \right] / \left[ 2 \left( \text{Im}(x)^2 + (\text{Re}(x) - 1)^2 \right)^{3/2} \right] \]

Ugh!

\textbf{Simplify} [%\]

\[-(x - 1) \left( (\text{Re}(x) - 1) \left( \sqrt{\text{Im}(x)^2 + (\text{Re}(x) - 1)^2} - \text{Re}(x) \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) \right) \right. - \left. \right. \left( \text{Im}(x)^2 \left( \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) - \log \left( \sqrt{\text{Im}(x)^2 + (\text{Re}(x) - 1)^2} + \text{Re}(x) - 1 \right) + \log \left( \sqrt{\text{Im}(x)^2 \left( 2 \text{Re}(x)^2 - 2 \text{Re}(x) + 1 \right) + \text{Im}(x)^4 + (\text{Re}(x) - i)^2 \text{Re}(x)^2} \right) + \text{Im}(x)^2 + \text{Re}(x)^2 - \text{Re}(x) \right) \right) \right] / \left[ 2 \left( \text{Im}(x)^2 + (\text{Re}(x) - 1)^2 \right)^{3/2} \right] \]

To see if this is correct, try to evaluate it at \(x = 1\) and see if we get \((1/2) \, 1 \, |1| = 1/2\):
It appears that Mathematica has created an expression involving the variable \( x \) that it can't evaluate for any particular value of \( x \! 

Human-computer collaboration on a definite integral

How does Mathematica do with that \( u = x \ln x \) example?

\[
\int_{0.9}^{1.1} \left( \log(x) + 1 \right) \sqrt{1 - x^2 \log^2(x)} \, dx
\]

It takes Mathematica half a minute or so to find this (useless) answer.

When we ask Mathematica to do this integral numerically, we get an answer quickly.

\[
\text{N[]}[
\]

0.199331

And this seems to be right, if we apply the Evaluation Theorem using the antiderivative we found before:

\[
\frac{1}{2} \left( \text{ArcSin}[x \log(x)] + x \log[x] \sqrt{1 - x^2 \log^2(x)} \right) /. x \to 1.1
\]  
\[
\frac{1}{2} \left( \text{ArcSin}[x \log(x)] + x \log[x] \sqrt{1 - x^2 \log^2(x)} \right) /. x \to 0.9
\]

0.199331

For some ideas about how did Mathematica might have done the calculation of the definite integral (without knowledge of the antiderivative), see section 6.5 of Stewart, which discusses numerical methods of integration.