Math 192r, Problem Set #10: Solutions

1. *Using a sign-reversing involution, prove that for all $n > k$, the sum $\sum_{m:\, k \le m \le n} s(n,m)S(m,k)$ equals zero.*

   (Note: Since for $n = k$ the sum is clearly 1 and for $n < k$ the sum is clearly 0, the above assertion can be rewritten as

   $$\sum_{m:\, k \le m \le n} s(n,m)S(m,k) = 1_{\{n=k\}},$$

   which is just the assertion that the s-matrix and S-matrix are inverses of one another.)

   Define a "gizmo" of type (n,m,k) as a permutation of $[n]$ with m cycles, along with a partition of those m cycles into k blocks, and assign it weight $(-1)^{n-m}$. Since the number of permutations of $[n]$ with m cycles is $(-1)^{n-m}s(n,m)$ and the number of partitions of m things into k blocks is $S(m,k)$, the sum $\sum_{m:\, k \le m \le n} s(n,m)S(m,k)$ is the sum of the weights of all gizmos of type (n,m,k), with m varying between k and n (inclusive).

   Say a block in some gizmo γ is ample if it involves more than one element of $[n]$; that is, it either contains more than one cycle or it involves a cycle of size greater than 1 (or both). Since $n > k$, there is at least one ample block in γ. Let x be the smallest element of $[n]$ that lies in an ample block of γ, and let y be the second-smallest element in this block.

   If x and y are in the same cycle, say as $(xab\ldots dyef\ldots h)$, replace that cycle by the pair of cycles $(xab\ldots d)$ and $(yef\ldots h)$, calling the resulting gizmo γ'. Note that the block of γ' that contains these two new cycles is ample, that x is the smallest element of $[n]$ that lies in an ample block of γ', and that y is the second-smallest element in this block.

   On the other hand, if x and y are in different cycles, say as $(xab\ldots d)$ and $(yef\ldots h)$, replace that pair of cycles by $(xab\ldots dyef\ldots h)$, calling the resulting gizmo γ'. Note that the block of γ' that contains the merged cycle is ample, that x is the smallest element of $[n]$ that lies in an ample block of γ', and that y is the second-smallest element in this block.

We see that $\Phi : \gamma \mapsto \gamma'$ is an involution. Also, when γ has m cycles, γ' has $m \pm 1$ cycles, so that Φ is sign-reversing. Hence the sum of the weights of all the gizmos of type (n, m, k) (with n, k fixed and satisfying $n > k$ and with m varying between k and n inclusive) is zero.

2. *Consider the subset of the square grid bounded by the vertices* $(0, 0)$, $(m, 0)$, $(0, n)$, *and* (m, n), *and let* q *be a formal indeterminate. Let the weight of the horizontal grid-edge joining* (i, j) *and* $(i + 1, j)$ *be* q^j *(for all* $0 \leq i \leq m - 1$ *and* $0 \leq j \leq n$*), and let each vertical grid-edge have weight 1. Define the weight of a lattice path of length* $m + n$ *from* $(0, 0)$ *to* (m, n) *to be the product of the weights of all its constituent edges. Let* $P(m, n)$ *be the sum of the weights of all the lattice paths of length* $m + n$ *from* $(0, 0)$ *to* (m, n)*, a polynomial in* q*. (Note that putting* $q = 1$ *turns* $P(m, n)$ *into the number of lattice paths of length* $m + n$ *from* $(0, 0)$ *to* (m, n)*, which is the binomial coefficient* $\frac{(m+n)!}{m!n!}$*.)*

   (a) *Give a formula for* $P(1, n)$ *and for the generating function*

   $$\sum_{n \geq 0} P(1, n) x^n.$$

   The $n + 1$ paths from $(0, 0)$ to $(1, n)$ have weight 1, q, q^2, $\ldots$, q^n, so $P(1, n) = 1 + q + q^2 + \ldots + q^n = (1 - q^{n+1})/(1 - q)$ and

   $$\begin{aligned} \sum_{n \geq 0} P(1, n) x^n &= \sum_{n \geq 0} \frac{x^n - q^{n+1} x^n}{1 - q} \\ &= \sum_{n \geq 0} \left(\frac{x^n}{1 - q} - \frac{q (qx)^n}{1 - q} \right) \\ &= \frac{1}{(1 - q)(1 - x)} - \frac{q}{(1 - q)(1 - qx)} \\ &= \frac{1}{(1 - x)(1 - qx)} \end{aligned}$$

   (b) *Find (and justify) a recurrence relation relating the polynomials* $P(m, n)$, $P(m - 1, n)$, *and* $P(m, n - 1)$ *that generalizes the Pascal triangle relation for binomial coefficients.*

A path p from $(0,0)$ to (m,n) passes through either $(1,0)$ or $(0,1)$, but not both.

In the first case, let p' be the path from 0 to $(m-1,n)$ obtained from p by snipping out the step from $(0,0)$ to $(1,0)$ and sliding the rest of the path one step to the left. In this case the weight of p' equals the weight of p.

In the second case, let p' be the path from 0 to $(m,n-1)$ obtained from p by snipping out the step from $(0,0)$ to $(0,1)$ and sliding the rest of the path one step downward. In this case the weight of p' equals the weight of p times q^m (since each of the m horizontal edges of p has weight equal to q times the weight of the corresponding horizontal edge of p').

Combining, we find that

$$P(m,n) = P(m-1,n) + q^m P(m,n-1).$$

Indeed, we can check this against (a), using the trivial case $P(0,n) = 1$: the relation $P(1,n) = P(0,n) + q^1 P(1,n-1)$ then amounts to $1 + q + q^2 + \ldots + q^n = 1 + q(1 + q + \ldots + q^{n-1})$, which is true.

(c) *Let $F_m(x)$ denote $\sum_{n\geq 0} P(m,n)x^n$. Use your answer from (b) to give a formula for $F_m(x)$ in terms of $F_{m-1}(x)$, and from this derive a non-recursive formula for $F_m(x)$.*

Multiply the inset equation by x^n and sum over all $n \geq 1$ (noting that the omitted term $P(m,0)x^0$ is just 1):

$$F_m(x) - 1 = (F_{m-1}(x) - 1) + q^m x F_m(x)$$

This gives $(1 - xq^m)F_m(x) = F_{m-1}(x)$ so that

$$F_m(x) = F_{m-1}(x)/(1 - xq^m).$$

Indeed, using this relation and the base case $F_0(x) = 1/(1-x)$ we get the general formula

$$F_m(x) = \frac{1}{(1-x)(1-xq)\cdots(1-xq^m)}.$$

(d) *Write a computer program to compute the polynomial $P(m,n)$ for any input values m,n.*

```
readlib(coeftayl);
F := proc(m) local i; product(1/(1-q^i*x),i=0..m); end;
P := proc(m,n) simplify(coeftayl(F(m),x=0,n)); end;
r := proc(m,n)
    simplify(P(m,n)*P(m-1,n-1)/P(m-1,n)/P(m,n-1)); end;
```

Note that the function `coeftayl` is a good thing to use when you want just one coefficient from a Taylor expansion; it saves time.

(e) *Compute $P(m,n)/P(m-1,n)$ for various values of $m \geq 1$ and $n \geq 0$ and conjecture a formula for it. Do the same for the ratio $P(m,n)/P(m,n-1)$ with $m \geq 0$ and $n \geq 1$.*

A bit of playing shows that the first ratio equals

$$(1+q+q^2+\ldots+q^{m+n-1})/(1+q+q^2+\ldots+q^{m-1})$$

or $(1-q^{m+n})/(1-q^m)$, while the second ratio equals or $(1-q^{m+n})/(1-q^n)$. That is, we conjecture that

$$P(m,n)/P(m-1,n) = (1-q^{m+n})/(1-q^m)$$

for $m \geq 1$ and $n \geq 0$ and

$$P(m,n)/P(m,n-1) = (1-q^{m+n})/(1-q^n)$$

for $m \geq 0$ and $n \geq 1$.

(f) *Use the recurrence relation from (b) to verify your conjectures from (e).*

We prove the two conjectures simultaneously by joint induction on m, n. That is, we verify the first formula for its base cases (where $m = 1$ or $n = 0$) and the second formula for its base cases (where $m = 0$ or $n = 1$), and we then verify that both formulas hold for m, n if both formulas are both assumed to hold when m, n are replaced by integers m', n' satisfying $m' + n' < m_n$.

We have $P(0,n) = 1$ and $P(1,n) = (1-q^{n+1})/(1-q)$ for all $n \geq 0$, so $P(1,n)/P(0,n) = (1-q^{1+n})/(1-q^1)$ for all $n \geq 0$ and $P(m,0)/P(m-1,0) = (1-q^{m+0})/(1-q^m)$ for all $m \geq 1$, as claimed. It is also easy to check that $P(m,0) = 1$ and

$P(m,1) = (1-q^{m+1})/(1-q)$ for all $m \geq 0$, so $P(0,n)/P(0,n-1) = (1-q^{0+n})/(1-q^m)$ for all $n \geq 1$ and $P(m,1)/P(m,0) = (1-q^{m+1})/(1-q^1)$ for all $m \geq 0$, as claimed.

Next we use the induction hypothesis to prove $P(m,n)/P(m-1,n) = (1-q^{m+n})/(1-q^m)$. Rewrite this as $P(m,n) = P(m-1,n)(1-q^{m+n})/(1-q^m)$. We know (from (b)) that $P(m,n) = P(m-1,n) + q^m P(m,n-1)$, so the thing we're trying to prove can be rewritten as $P(m-1,n)(1-q^{m+n})/(1-q^m) = P(m-1,n) + q^m P(m,n-1)$ or

$$P(m-1,n)(1-q^n)/(1-q^m) = P(m,n-1).$$

But by the induction hypothesis we have $P(m-1,n) = P(m-1,n-1)(1-q^{m-1+n})/(1-q^n)$ and $P(m,n-1) = P(m-1,n-1)(1-q^{m+n-1})/(1-q^m)$; dividing the first of these by the second gives us what we need to prove.

Lastly we use the induction hypothesis to prove $P(m,n)/P(m,n-1) = (1-q^{m+n})/(1-q^n)$. Rewrite this as $P(m,n) = P(m,n-1)(1-q^{m+n})/(1-q^n)$. We know (from (b)) that $P(m,n) = P(m-1,n) + q^m P(m,n-1)$, so the thing we're trying to prove can be rewritten as $P(m,n-1)(1-q^{m+n})/(1-q^n) = P(m-1,n) + q^m P(m,n-1)$ or

$$P(m,n-1)(1-q^m)/(1-q^n) = P(m-1,n),$$

which we proved in the preceding paragraph.

This completes the proof.

Note that we can conclude as a corollary that $P(m,n) = P(n,m)$ for all m,n (though as we'll see there are easier ways to prove this!).