1. Use the recurrence for $p(n)$ to compute the last digit of $p(n)$ for every $n$ between 1 and 1000. Can you make any conjectures about the relationship between the last digit of $n$ and the last digit of $p(n)$?

Here’s a Maple program that does this:

```maple
F := proc(n) option remember; local total, k;
if n=0 then 1; elif n<0 then 0; else total := 0;
  k := 1; while k*(3*k+1)/2 <= n do
    total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k+1: od:
  k := -1; while k*(3*k+1)/2 <= n do
    total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k-1: od:
  total mod 10; fi: end;
```

We then create a matrix to keep track of how often it happens that $n$ ends with the digit $i$ while $p(n)$ ends with the digit $j$ (for $i, j$ between 0 and 9), and print out its entries:

```maple
for i from 0 to 9 do for j from 0 to 9 do a[i,j]:=0: od: od:
for n from 1 to 1000 do k := F(n);
  a[n mod 10, k mod 10] := a[n mod 10, k mod 10] + 1; od:
for i from 0 to 9 do seq(a[i,j],j=0..9) od;
```

This results in the output

```
14, 7, 13, 12, 3, 5, 12, 15, 9, 10
9, 11, 14, 9, 8, 9, 10, 13, 7, 10
3, 14, 12, 14, 10, 8, 8, 12, 6, 13
8, 9, 12, 9, 8, 17, 5, 13, 9, 10
49, 0, 0, 0, 0, 51, 0, 0, 0, 0
5, 12, 10, 4, 15, 7, 13, 13, 8, 13
8, 14, 11, 15, 7, 9, 8, 6, 5, 17
9, 10, 6, 9, 16, 10, 9, 9, 12, 10
10, 9, 16, 8, 11, 11, 11, 13, 5, 6
48, 0, 0, 0, 0, 52, 0, 0, 0, 0
```

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from which we conjecture that when \( n \) ends in 4 or 9, \( p(n) \) ends in 0 or 5. That is, if \( n \) is 1 less than a multiple of 5, \( p(n) \) is a multiple of 5. (This fact was first noticed and proved by Ramanujan.)

2. Let \( F(0) = 1 \) and recursively define \( F(n) = F(n - 1) + F(n - 3) - F(n - 6) - F(n - 10) + F(n - 15) + F(n - 21) \ldots \) for all \( n > 0 \), where terms of the form \( F(n - k) \) are to be ignored once \( k > n \). There exists a set \( S \) of positive integers such that \( F(n) \) equals the number of partitions of \( n \) into parts belonging to \( S \). Find \( S \) (conjecturally).

This property of \( S \) is equivalent to 
\[
1 - q + q^3 - q^6 + q^{10} - + + + \ldots = \prod_{k \in S}(1 - q^k).
\]
We note first that we must have \( 1 \in S \), since if not, every factor in the product would have vanishing coefficient of \( q^1 \). Next we have 
\[
\prod_{k \in S \setminus \{1\}}(1 - q^k) = (1 - q - q^3 + \ldots)/(1 - q) = 1 + 0q + 0q^2 - q^3\ldots
\]
We now note that we must have \( 2 \not\in S \), since otherwise we would have 
\[
\prod_{k \in S \setminus \{1\}}(1 - q^k) \text{ of the form } 1 - q^2\ldots\]
On the other hand, we must have \( 3 \in S \). We now continue with 
\[
\prod_{k \in S \setminus \{1, 3\}}(1 - q^k) = (1 - q - q^3 + q^6\ldots)/(1 - q)(1 - q^3) = 1 + 0q + 0q^2 + 0q^3 - q^4 - q^5 - q^7\ldots
\]
To conclude that \( 4 \) and \( 5 \) (but not \( 6 \)) belong to \( S \). And so on. Empirically, we find that \( S \) is just the set of numbers that are not congruent to \( 2 \mod 4 \).