Math 192r, Problem Set #15: Solutions

1. *Using the combinatorial definition of the determinant, prove that for all n-by-n matrices A, B,* $\det(AB) = \det(A)\det(B)$.

   Let $C = AB$, so that $C_{i,k} = \sum_j A_{i,j} B_{j,k}$. Then

$$\det(C) = \sum_\pi \mathrm{sign}(\pi) \sum_\mu \prod_i A_{i,\mu(i)} B_{\mu(i),\pi(i)},$$

   where π ranges over all permutations of $\{1, ...n\}$ and μ ranges over all mappings from $\{1, ..., n\}$ to itself. Let's interchange the order of summations. We'll show that for each fixed μ, the sum

$$\sum_\pi \mathrm{sign}(\pi) \prod_i A_{i,\mu(i)} B_{\mu(i),\pi(i)}$$

   vanishes unless μ is a permutation. Suppose that μ is not a permutation; then $\mu(i_1) = \mu(i_2)$ for some $i_1 \neq i_2$. For notational simplicity and definiteness, assume $i_1 = 1$ and $i_2 = 2$ and write $m = \mu(1) = \mu(2)$. Let τ be the transposition that swaps 1 and 2. Then it is easy to check that the map $\pi \mapsto \pi' = \tau \circ \pi$ is a weight-reversing involution on the terms of the sum. Specifically, $\mu(1) = m = \mu(2)$ implies

$$\begin{aligned}
\prod_{i=1}^{2} A_{i,\mu(i)} B_{\mu(i),\pi(i)} &= A_{1,m} B_{m,\pi(1)} A_{2,m} B_{m,\pi(2)} \\
&= A_{1,m} B_{m,\pi(2)} A_{2,m} B_{m,\pi(1)} \\
&= A_{1,m} B_{m,\pi'(1)} A_{2,m} B_{m,\pi'(2)} \\
&= \prod_{i=1}^{2} A_{i,\mu(i)} B_{\mu(i),\pi'(i)},
\end{aligned}$$

   and since $\pi(i) = \pi'(i)$ for all $i > 2$ we have

$$\prod_{i=3}^{n} A_{i,\mu(i)} B_{\mu(i),\pi(i)} = \prod_{i=3}^{n} A_{i,\mu(i)} B_{\mu(i),\pi'(i)}.$$

   Combining these facts, we have

$$\prod_{i=1}^{n} A_{i,\mu(i)} B_{\mu(i),\pi(i)} = \prod_{i=1}^{n} A_{i,\mu(i)} B_{\mu(i),\pi'(i)}.$$

But $\operatorname{sign}(\pi) = -\operatorname{sign}(\pi')$, so the two terms cancel.

$\det(C)$ is therefore equal to

$$\sum_\pi \operatorname{sign}(\pi) \sum_\rho \prod_i A_{i,\rho(i)} B_{\rho(i),\pi(i)}.$$

Letting $\sigma = \pi \circ \rho^{-1}$ (so that $\pi = \sigma \circ \rho$), we can write

$$\begin{aligned}
\prod_i \left(A_{i,\rho(i)} B_{\rho(i),\pi(i)}\right) &= \left(\prod_i A_{i,\rho(i)}\right)\left(\prod_i B_{\rho(i),\pi(i)}\right) \\
&= \left(\prod_i A_{i,\rho(i)}\right)\left(\prod_i B_{i,\sigma(i)}\right) \\
&= \prod_i \left(A_{i,\rho(i)} B_{i,\sigma(i)}\right),
\end{aligned}$$

so that $\det(C)$ equals

$$\sum_\rho \sum_\sigma \operatorname{sign}(\rho)\operatorname{sign}(\sigma) \prod_i A_{i,\rho(i)} B_{i,\sigma(i)}.$$

But this factors as

$$\sum_\rho \operatorname{sign}(\rho) \prod_i A_{i,\rho(i)}$$

times

$$\sum_\sigma \operatorname{sign}(\sigma) \prod_i B_{i,\sigma(i)},$$

or $\det(A)$ times $\det(B)$.

2. *Use Lindstrom's lemma, the interpretation of domino tilings as routings, and a computer, in order to count the domino tilings of an 8-by-8 square, as well as the domino tilings of an 8-by-8 square from which two (non-opposite) corners have been removed.*

   Checkerboard-color the squares in the grid, so that the upper-left square is shaded. Mark the mid-point of every vertical edge that has a black square to its left or a white square to its right (or both). It's easy to check that every possible placement of a domino yields either zero or two marked points on its boundary. Hence, if one fixes a domino tiling and draws connections between all pairs of marked points that share a domino, one gets four non-intersecting left-to-right lattice paths joining the four leftmost marked points to the four rightmost marked points. Conversely, given four such lattice paths, one can construct a tiling by taking all those dominoes that cover an edge of the lattice path, along with all dominoes that are centered on those marked points that do not lie on any of the lattice paths. Hence there is a bijection between domino-tilings of the 8-by-8 grid and families of non-intersecting lattice paths joining the sources s_1, s_2, s_3, s_4 to the sinks t_1, t_2, t_3, t_4 in a trellis-like directed graph, with directed edges corresponding to the vectors $(1,1)$, $(1,-1)$, and $(2,0)$. It is easy to see that the only way to connect the s_i's and the t_j's via non-intersecting paths in this directed graph is to connect s_i to t_i for $1 \le i \le 4$. Hence Lindstrom's Lemma applies, and the number of families of non-intersecting lattice paths is equal to the determinant of the 4-by-4 matrix M whose i,jth entry equals the number of lattice paths from s_i to t_j.

To determine the entries of M, we introduce new vertices in a shifted lattice that fills the holes in the lattice of marked points. (That is to say, we now associated a point with every vertical edge.) The points s_1, s_2, s_3, s_4 are the 2nd, 4th, 6th, and 8th points on the left edge (and similarly for t_1, t_2, t_3, t_4). Then the i,jth entry of M is equal to the $2i, 2j$th entry of $AA^TAA^TAA^TAA^T$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using Maple, one gets

```
[ 22    68    30    48    10    12    1     1  ]
[                                              ]
[ 68   236   116   216    60    84    13    14 ]
[                                              ]
[ 30   116    62   128    41    61    11    12 ]
```

```
[                                              ]
[ 48  216  128  320  129  230   60   70 ]
[                                              ]
[ 10   60   41  129   63  128   40   48 ]
[                                              ]
[ 12   84   61  230  128  306  116  146 ]
[                                              ]
[  1   13   11   60   40  116   52   68 ]
[                                              ]
[  1   14   12   70   48  146   68   90 ]
```

Extracting the sub-matrix

```
[ 236  216   84   14 ]
[                    ]
[ 216  320  230   70 ]
[                    ]
[  84  230  306  146 ]
[                    ]
[  14   70  146   90 ]
```

and taking its determinant, one gets 12988816.

To solve the other part of the problem, in which two non-opposite corners (say the two lower corners) get removed, one can just get rid of s_4 and t_4, obtaining thereby the three-by-three matrix

```
[ 236  216   84 ]
[               ]
[ 216  320  230 ]
[               ]
[  84  230  306 ]
```

whose determinant is 2436304.