

Math 192r, Problem Set #19: Solutions

1. In problem #3 of assignment #17, multivariate polynomials

$$D(x_1, x_3, \dots, x_{2n+1}; y_2, y_4, \dots, y_{2n})$$

were defined. Find an infinite acyclic directed graph with special vertices $\dots, v_{-1}, v_0, v_1, \dots$ where all edges are assigned weight 1 and vertices are assigned weights according to some scheme that you must devise, so that for all integers $i \leq j$ the sum of the weights of the paths from v_i to v_j is $D(x_{2i+1}, x_{2i+3}, \dots, x_{2j+1}; y_{2i+2}, y_{2i+4}, \dots, y_{2j})$. Include a proof that your answer is correct.

We can build a model of our directed graph in \mathbf{Z}^2 . Our vertices will be pairs (i, j) with $i - j$ equal to 0, 1, 2, or 3, and there will be a directed edge from vertex (i, j) to vertex (i', j') if and only if $(i' - i, j' - j) = (1, 0)$ or $(0, 1)$. The special vertices are $v_i = (i, i)$. The weight of vertex (i, j) is x_{i+j} if $i - j = 0$, $y_{i+j}/x_{i+j-1}x_{i+j+1}$ if $i - j = 1$, $x_{i+j}/y_{i+j-1}y_{i+j+1}$ if $i - j = 2$, and y_{i+j} if $i - j = 3$. Let $W_{i,j}$ be the sum of the weights of the paths from v_i to v_j . It is easy to check that $W_{i,i} = x_{2i}$ and $W_{i,i+1} = y_{2i+1}$ as required. To prove that $W_{i,j} = D(x_{2i+1}, \dots, x_{2j+1}; y_{2i+2}, \dots, y_{2j})$ for all values of i and j , it suffices to verify that the numbers $W_{i,j}$ satisfy the diamond recurrence. That is, we must show that $W_{i,j}W_{i+1,j-1} - W_{i+1,j}W_{i,j-1} = 1$. By Lindstrom's lemma, this is equivalent to the assertion that the signed sum of the weights of the 2-routings from $\{v_i, v_{i+1}\}$ to $\{v_j, v_{j-1}\}$ is 1. But we already know that there is a unique 2-routing of this kind, connecting v_i to v_j and v_{i+1} to v_{j-1} , so it suffices to check that the weight of this 2-routing is 1. To check this, note that the product of the weights of (i, i) , $(i+1, i)$, $(i+1, i+1)$, $(i+2, i)$, $(i+2, i+1)$, and $(i+2, i+2)$ is $(x_{2i})(y_{2i+1}/x_{2i}x_{2i+2})(x_{2i+2})(x_{2i+2}/y_{2i+1}y_{2i+3})(y_{2i+3}/x_{2i+2}x_{2i+4})(x_{2i+4}) = 1$, the product of the weights of $(i+3, i)$, $(i+3, i+1)$, $(i+3, i+2)$, and $(i+3, i+3)$ is $(y_{2i+3})(x_{2i+4}/y_{2i+3}y_{2i+5})(y_{2i+5}/x_{2i+4}x_{2i+6})(x_{2i+6}) = 1$, the product of the weights of $(i+4, i+1)$, $(i+4, i+2)$, $(i+4, i+3)$, and $(i+4, i+4)$ is 1, etc.

2. Consider an infinite array with tilted upper boundary like the one shown

below:

$$\begin{array}{cccccccc}
 & & & & & & & \vdots \\
 & & & & & & & x_5 \\
 & & & & & & x_4 & w_5 & y_5 \\
 & & & & x_3 & & w_4 & & y_4 & * \\
 & & & x_2 & w_3 & & y_3 & & * & * \\
 & & x_1 & w_2 & & y_2 & * & * & * & * \\
 w_1 & & y_1 & * & * & * & * & * & * & * \\
 \vdots & & & & & & & & & \vdots
 \end{array}$$

Here the entries w_i, x_i, y_i are formal indeterminates, and the entries marked with asterisks are determined by the diamond rule as in assignment #17; that is, whenever the array contains four entries arranged like

$$\begin{array}{ccc}
 & a & \\
 b & & c \\
 & d &
 \end{array}$$

we must have $ad - bc = 1$. Some experimentation will probably convince you that each entry in the table is a Laurent polynomial in the variables w_i, x_i, y_i , and that moreover each coefficient in this polynomial equals $+1$. Show how for each such Laurent polynomial, the Laurent monomials that participate correspond to the perfect matchings of some graph (just as was the case in assignment #17). Give a concrete description of the graphs and the correspondence between matchings and monomials (including either a proof or convincingly large examples).

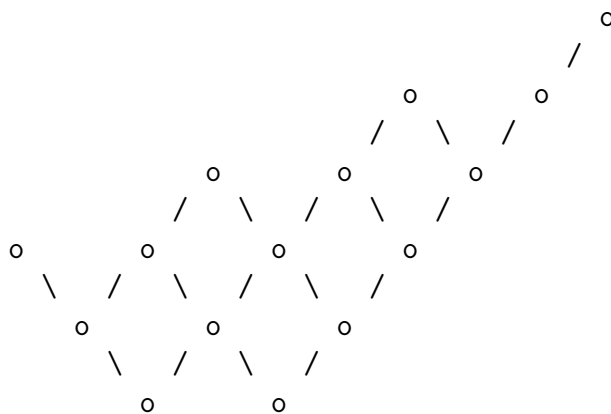
Consider first what we get when we specialize all the variables to equal 1:

$$\begin{array}{cccccccc}
 & & & & & & & \vdots \\
 & & & & & & & 1 \\
 & & & & & & \mathbf{1} & 1 & 1 \\
 & & & & 1 & 1 & 1 & 2 & 7 \\
 & & & \mathbf{1} & 1 & 1 & 2 & 3 & 7 \\
 & & 1 & 1 & 1 & 2 & 3 & 7 & 11 \\
 1 & & 1 & 2 & 3 & 7 & 11 & 26 & 41 \\
 \vdots & & & & & & & & \vdots
 \end{array}$$

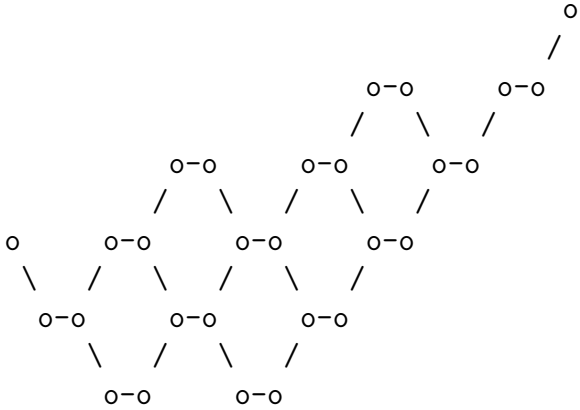
I claim that each of these entries counts the number of lattice paths joining two points on the upper boundary of 1's, where a lattice path may pass only through vertices that are marked with 1's, 2's, 3's, and 7's (call these the "single-digit locations"). Specifically, given a location in the table, trace a diagonal going northwest until you hit the boundary, and call that location i ; likewise, trace a diagonal going northeast until you hit the boundary, and call that location j . Then I claim that the entry in question is equal to the number (call it $N(i, j)$) of paths from location i to location j consisting of northeast and southeast steps, by way of single-digit locations.

For example, I claim that the 3 in the last row of the above excerpt counts the lattice paths between the two boldface 1's on the boundary. To see why this is true, take four entries that form a diamond, and let the associated locations on the boundary be i, i', j', j (from left to right). To get a proof by induction that the entries count lattice paths, we need to verify that $N(i, j)N(i', j') - N(i, j')N(i', j) = 1$. But (by Lindstrom's lemma) this is a consequent of the fact that there is a unique 2-routing that joins $\{i, i'\}$ with $\{j, j'\}$, and that it connects i with j and i' with j' . (That's because the single-digit locations in any column are precisely the top two locations in that column.)

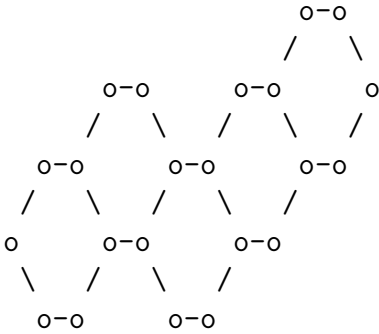
We have now shown that the entries in the table count lattice paths. But a lattice path is just a special case of a routing (namely, a 1-routing), so by using the routings-into-matchings trick we can get a bijection between lattice paths in that graph and perfect matchings of a certain graph. For instance, consider the directed graph



The 11 lattice paths joining the leftmost and rightmost vertices in this directed graph correspond to the 11 perfect matchings of the graph

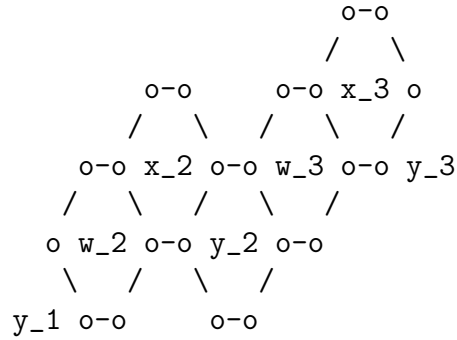


which in turn correspond to the 11 perfect matchings of the graph

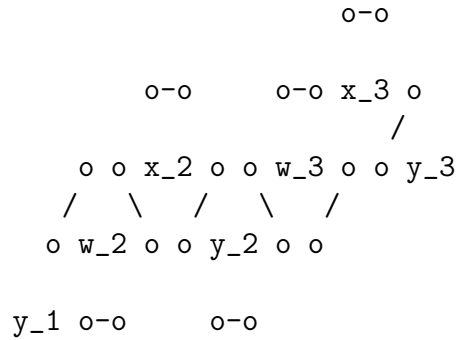


Now we may bring the variables w_i, x_i, y_i back into the story. If indeed the rational functions that are generated by the two-dimensional recurrence are Laurent polynomials, and if indeed each monomial contributing to these polynomials has coefficient $+1$, then we can say that the number of monomials must go like 1, 1, 1, 2, 3, 7, 11, 26, 41, \dots , and they should correspond to matchings of the above graph.

To see how the correspondence goes, label the hexagonal cells as shown in the diagram on the next page:



(Note that we've added two extra "ghost-cells" at the ends.) Given a matching μ of the graph (and its associated monomial) and a cell c in the graph (and its associated variable), the exponent of the variable in the monomial is equal to 2 minus the number of edges in μ adjacent to c , unless c is a ghost-cell, in which case the exponent of the variable in the monomial is equal to 1 minus the number of edges in μ adjacent to c . For instance, the matching



corresponds to the monomial

$$y_1^{1-0} w_2^{2-3} x_2^{2-3} y_2^{2-3} w_3^{2-3} x_3^{2-2} y_3^{1-1} = \frac{y_1}{w_2 x_2 y_2 w_3}.$$