Implication

Logical implication (sometimes called “material implication”) is counter-intuitive. Regardless of the nature of the propositions \( p \) and \( q \), we say that the compound proposition \( p \rightarrow q \) is true when \( p \) and \( q \) are both true, or when \( p \) and \( q \) are both false, or when \( p \) is false and \( q \) is true; the only case in which we deem \( p \rightarrow q \) to be false is when \( p \) is true and \( q \) is false.

A compact (but somewhat confusing) way to express \( p \rightarrow q \) is \( \neg p \lor q \). That is, the most efficient way to implement \( \rightarrow \) as a circuit uses one NOT-gate and one OR-gate.

A different way to represent \( p \rightarrow q \) is to use “disjunctive normal form”, and to write \( p \rightarrow q \) as a disjunction of conjunctions, namely,

\[
(p \land q) \lor (\neg p \land \neg q) \lor (\neg p \land q).
\]

Notice that this is the definition of \( p \rightarrow q \) given in the first paragraph, rendered symbolically.

What about \( p \leftarrow q \)? \( p \leftarrow q \) is equivalent to \( q \rightarrow p \), which is equivalent to \( \neg q \lor p \), which is equivalent to \( p \lor \neg q \).

Finally, what about \( p \leftrightarrow q \)? One way to write it is as \( (p \rightarrow q) \land (p \leftarrow q) \). Since \( p \rightarrow q \) can be expressed as \( \neg p \lor q \), and \( p \leftarrow q \) can be expressed as \( p \lor \neg q \), we can rewrite \( p \leftrightarrow q \) as the conjunction \( (\neg p \lor q) \land (p \lor \neg q) \), which in terms of circuitry uses two NOT gates, two OR gates, and one AND gate. Notice that this is the conjunctive normal form of \( p \leftrightarrow q \), expressing it as a conjunction of disjunctions.

Alternatively, we can represent \( p \leftarrow q \) in disjunctive normal form as \( (p \land q) \lor (\neg p \land \neg q) \), using two NOT gates, two AND gates, and one OR gate.