Implication

Logical implication (sometimes called "material implication") is counter-intuitive. Regardless of the nature of the propositions p and q, we say that the compound proposition $p \to q$ is true when p and q are both true, or when p and q are both false, or when p is false and q is true; the only case in which we deem $p \to q$ to be false is when p is true and q is false.

A compact (but somewhat confusing) way to express $p \to q$ is $\neg p \vee q$. That is, the most efficient way to implement $\to$ as a circuit uses one NOT-gate and one OR-gate.

A different way to represent $p \to q$ is to use "disjunctive normal form", and to write $p \to q$ as a disjunction of conjunctions, namely,

$$(p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q).$$

Notice that this is the definition of $p \to q$ given in the first paragraph, rendered symbolically.

What about $p \leftarrow q$? $p \leftarrow q$ is equivalent to $q \to p$, which is equivalent to $\neg q \vee p$, which is equivalent to $p \vee \neg q$.

Finally, what about $p \leftrightarrow q$? One way to write it is as $(p \to q) \wedge (p \leftarrow q)$. Since $p \to q$ can be expressed as $\neg p \vee q$, and $p \leftarrow q$ can be expressed as $p \vee \neg q$, we can rewrite $p \leftrightarrow q$ as the conjunction $(\neg p \vee q) \wedge (p \vee \neg q)$, which in terms of circuitry uses two NOT gates, two OR gates, and one AND gate. Notice that this is the conjunctive normal form of $p \leftrightarrow q$, expressing it as a conjunction of disjunctions.

Alternatively, we can represent $p \leftarrow q$ in disjunctive normal form as $(p \wedge q) \vee (\neg p \wedge \neg q)$, using two NOT gates, two AND gates, and one OR gate.