Some words about words

Here is some information about some math words, including a few whose use in mathematics is different from their use in ordinary language. If later in the semester you trip across any examples of misleading or obscure math-language that the book doesn't explain well, and you wish that this document had discussed them, let me know, and I'll include them in future versions of this document. (You'll be doing future students a favor **and** you'll get course engagement credit!)

1. **Zero versus the empty set**

Zero is a *number*; the empty set is a *set* with zero elements. Both are handy in similar ways. It's handy to have zero as a *number* so that you can ask "How many elements does this set have?" before you know whether the set actually has any. Likewise, it's handy to have the empty set as a *set* so that you can, for instance, talk about the intersection of two sets before you know whether the two sets actually intersect. But zero and the empty set are not the same thing. In this class, please write zero as "0" (an oval without a slash) and the empty set as "$\varnothing$" (a circle with a slash) so that it's clear which one you mean. Or you could use set-builder notation and write the empty set as $\{\,\}$.

(Fun fact: In some work on what's called "foundations of math", zero gets defined as the empty set. Ditto for a number system called the *surreal numbers*. But we're not doing foundations of math or studying surreal numbers.)

2. **"Positive", "negative"**

A positive number is one that is greater than 0. A negative number is one that is less than 0. 0 is neither positive nor negative. (Fun fact: In France, at least in some contexts, zero is considered "positif". But we're not in France.)

3. **"Natural"**

In this class (and in most computer science contexts), 0 is considered a natural number (though some mathematicians consider 0 "unnatural" and insist that 1 is the first natural number).

4. **"Even", "Odd"**

Sure, dividing an empty set of cookies into two equally large empty sets doesn't really feel like dividing the empty set in two. But if we were to call zero an odd number, then it would be an odd number than remains odd when you add 1 to it. If we want the odd and even numbers to continue to alternate in a well-behaved way as we move ever further to the left on the number line, we'd better call 0 even, -1 odd, -2 even, and so on.

Here's a different way to look at it. What property do 2,4,6,... have in common that 1,3,5,... lack? When you divide them by two, the remainder is zero. When you divide zero by two, what remainder do you get? Zero! So 0 is even.

Or, if you prefer, look at the final digits of numbers. All the positive integers that end in 0,2,4,6,8 are even; all the positive integers that end in 1,3,5,7,9 are odd. What digit does 0 end in? 0. So that's another reason to regard 0 as even.

If you're still not convinced that 0 should be regarded as even rather than odd, then just regard this as part of the definition of the words "even" and "odd".

Better yet, intead of treating 0 as a special case, remember the *definition* of even and odd: An integer is even if it is divisible by 2 and odd if it isn't divisible by 2.

5. **"Iff"**

"Iff" means if-and-only-if. Example: "You are married if you are a husband" is true, but "You are married iff you are a husband" is false (roughly half of all married people are wives!).

6. **"Divides"**

When a and b are integers with b nonzero, "a divides b" means b is divisible by a, i.e., b/a is an integer. (For instance, 2 and 3 divide 6, but 4 doesn't.) Sometimes people say "a evenly divides b", but don't be misled (by the use of the word "evenly") into thinking that this has something to do with a or b or b/a being even.

Note that "a divides b" says that b divided by a (not a divided by b!) is an integer. Confusing, I know. Just be glad mathematicians don't say "a subtracts b" to mean a is greater than or equal to b or a is less than or equal to b.

When a divides b, we also say that a is a factor of b, or (turning things around) we say that b is a multiple of a.

By the way, even though 0 is smaller than all the numbers 1, 2, 3, ..., it's also divisible by all of them! The smallest natural number is a multiple of all the bigger natural numbers! Yes, zero is weird.

7. **"Let"**

In math, "Let A be a set with $|A| = n$" usually means "Let A be an arbitrary set with $|A| = n$." If the next sentence asks you to prove something about A, you can't just take A to be some particular set with n elements and prove that the claim holds in that one case.

By way of comparison, later on I might ask a question like "Let n be an odd number. Show that $n + 1$ must be even." You would NOT get credit if you wrote "Okay, I'll let $n = 3$, which is odd. Then $n + 1 = 4$, which is even." I would want to see an argument like this: "Suppose n is odd. Then n is of the form $2k + 1$. So then $n + 1$ is of the form $2k + 2$, which can be written as $2(k + 1)$, so it is even." (Here I'm assuming that at that point in the course you would know that a number is even iff it is a multiple of 2 and that a number is odd iff it is 1 more than a multiple of 2.)

8. **"Distinct" versus "unique"**

People new to writing math-prose often confuse these two words.

We say a mathematical entity is *unique* when there's nothing else like it. "There exists a unique real number x such that $x^2 = 0$" means there's only one such number.

We say that two mathematical entities are *distinct* when they are unequal to each other. "There exist distinct real numbers x, y such that $x^2 = y^2$" means you can find real numbers $x \neq y$ satisfying $x^2 = y^2$. More generally, we say that n mathematical entities are distinct when no two of them are equal.

Usually we apply the word "unique" to a single thing, and "distinct" to two or more things.

Here are a few (somewhat artificial) examples to highlight the difference between these two words:

"There exist distinct integers x, y such that $x^2 + y^2 = 1$" is true (just take $x = 0$ and $y = 1$, for instance; there are other solutions but just one will do to prove existence);

"There exist unique integers x, y such that $x^2 + y^2 = 1$" is false (there are four solutions $(x, y) = (0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$, though finding just two solutions suffices to prove non-uniqueness);

"There exist distinct integers x, y such that $x^2 + y^2 = 0$" is false; and

"There exist unique integers x, y such that $x^2 + y^2 = 0$" is true.

Moral: "distinct" and "unique" mean very different things.

A symbol we sometimes use to assert "There exists a unique x" is $\exists! x$; for instance, we write $(\exists! x)(\exists! y)\ x^2 + y^2 = 0$.

9. **"Prime", "composite", "relatively prime"**

A *prime* number is an integer greater than 1 that has exactly two positive divisors, namely 1 and itself; the primes are 2, 3, 5, 7, etc.

The integers greater than 1 that have more than two positive divisors (i.e., the ones that can be written in the form $a \times b$ where a and b are integers bigger than 1) are called *composite*.

The integer 1 is considered to be neither prime nor composite.

(Fun fact: In ancient Greece, 1 was considered not to be a number at all. Now of course we think of 1 as a number. Once 1 was accepted as a number, it was natural to think it was prime since it can't be factored, and some older definitions of the word "prime" do indeed apply to 1; but as number theory developed, treating 1 as a prime led to problems, and it was agreed that 1 should be considered to be neither prime nor composite. It took a while for the new definition of "prime" to be adopted universally, and as recently as the early 20th century there were tables of prime numbers that began with 1. But it is no longer the early 20th century.)

Two (not necessarily prime) numbers are called "relatively prime" when they have no common factor. So for instance 10 and 21 are relatively prime (to one another) even though neither of them is prime.

10. **Capital letters versus lower case letters**

Math needs lots of symbols. To increase the number of symbols available to mean different things, we distinguish between A and a, between B and b, and so on. So please don't thoughtlessly switch between capital and lower case. If a homework problem is about something called n don't rename it N, and vice versa.