

## Zero factorial

To start with an example, let  $A = \{1, 2\}$  and  $B = \{3, 4\}$ , so that  $A \cup B = \{1, 2, 3, 4\}$ , and note that  $\sum_{x \in A \cup B} x = 1 + 2 + 3 + 4 = (1 + 2) + (3 + 4) = \sum_{x \in A} x + \sum_{x \in B} x$ .

It seems reasonable that if  $A$  and  $B$  are disjoint finite sets of numbers, then  $\sum_{x \in A \cup B} x = \sum_{x \in A} x + \sum_{x \in B} x$ .

Likewise, if we let  $\prod_{x \in S} x$  denote the product of all the elements of the (finite) set  $S$ , it seems reasonable that if  $A$  and  $B$  are disjoint finite sets of numbers, then

$$\prod_{x \in A \cup B} x = \left( \prod_{x \in A} x \right) \left( \prod_{x \in B} x \right).$$

(E.g., with  $A = \{1, 2\}$  and  $B = \{3, 4\}$ ,  $\prod_{x \in A \cup B} x = 1 \times 2 \times 3 \times 4 = (1 \times 2)(3 \times 4) = (\prod_{x \in A} x) (\prod_{x \in B} x)$ .)

However, if we define  $\prod_{x \in \emptyset} x$  to be 0, this formula is going to fail; we need to take  $\prod_{x \in \emptyset} x$  to be 1. To see why, take  $A = \{1, 2\}$ ,  $B = \{\} = \emptyset$ . Then  $A \cup B = \{1, 2\} = A$ , so  $\prod_{x \in A \cup B} x$  and  $\prod_{x \in A} x$  both equal  $1 \times 2 = 2$ ; but then the formula above tells us that we must have  $\prod_{x \in B} x = (\prod_{x \in A \cup B} x) / (\prod_{x \in A} x) = 2/2 = 1$ .

This flies in the face of our knee-jerk tendency to associate the empty set with the number 0, but if we want formulas to hold as broadly as possible and not have needless exceptions, it's the right way to go.

One consequence of this is that  $0!$  should be defined to be 1, not 0:

$$\begin{aligned} 3! &= \prod_{x \in \{1, 2, 3\}} x = 1 \times 2 \times 3 = 6 \\ 2! &= \prod_{x \in \{1, 2\}} x = 1 \times 2 = 2 \\ 1! &= \prod_{x \in \{1\}} x = 1 \\ 0! &= \prod_{x \in \{\}} x = 1 \end{aligned}$$

A different way to explain why we want  $0!$  to equal 1 is to consider the formula  $n! = n(n-1)!$ . If we want it to be true for  $n = 1$ , we'd better set  $0!$  to be 1.

Also, if we want Theorem 2.2.8 to be true when  $n = k$ , we'd better define  $0!$  to equal to 1. Certainly defining  $0!$  to be 0 is no good; that'd give us 0 in the denominator!

It may also help you to compare the equation  $0! = 1$  with the equation  $x^0 = 1$  (valid for all nonzero  $x$ ). The first time you learned about exponents, you learned that  $x^n$  means  $x$  times  $x$  times ... times  $x$ , where the number of

$x$  you write down is  $n$ . Under this way of thinking,  $x^0$  means you write down zero  $x$ 's, so you might think that the product should be zero. But as we saw at the top of the page, the product of the elements of the empty set should be thought of as being 1, not 0.