The Fibonacci numbers

Example 12.5.1 and Exercise 12.5.1 ask you to apply matrix algebra to derive a formula for the kth Fibonacci number F_k. The derivation involves irrational numbers, and it can turn ugly if you let it. The trick to avoiding messy expressions (and to avoid the high probability of error that arises when you manipulate messy expressions) is to *introduce symbols* for the roots of the relevant quadratic equation and stick with them for as long as possible.

Specifically, the characteristic polynomial of

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is the determinant of

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix}$$

which is $(1-\lambda)(0-\lambda)-(1)(1) = \lambda^2-\lambda-1$, which factors as $(\lambda-\alpha)(\lambda-\beta)$ where α, β are distinct roots of $\lambda^2-\lambda-1=0$; say $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$.

There's a theorem that says that if an n-by-n matrix A has n distinct eigenvalues, then it's diagonalizable, and its diagonal form D is just the diagonal matrix with those n eigenvalues on the diagonal. Applying this theorem here in the case $n=2$, and using the fact that $\alpha \neq \beta$, we conclude that A is diagonalizable and that the diagonal form of A is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Let's find eigenvectors.

$\lambda = \alpha$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

which we solve to obtain $x = \alpha y$. So

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = \alpha$.

$\lambda = \beta$:
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta x \\ \beta y \end{pmatrix}$$
which we solve to obtain $x = \beta y$. So
$$\begin{pmatrix} \beta \\ 1 \end{pmatrix}$$
is an eigenvector for $\lambda = \beta$.

Thus we may take
$$P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix},$$
and we already have
$$D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$
Also recall the general formula
$$A^m = PD^mP^{-1}$$
(a consequence of $A = PDP^{-1}$), and remember the matrix equation
$$\begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = A^{k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
from Example 12.5.1. Putting this all together, we get
$$\begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{k-1} & 0 \\ 0 & \beta^{k-1} \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
I'll leave it to you to verify that this yields
$$F_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k).$$
At this point, if you want, you can replace α by $\alpha = (1+\sqrt{5})/2$ and β by $(1-\sqrt{5})/2$, obtaining the formula that appears in the book. But my point is that we gain a lot by using α and β throughout nearly all of the calculation.

Moral: When you solve a problem that involves lots of algebra, you're allowed to introduce your own symbol for a sub-expression that occurs a lot, so you don't have to tire your wrist by writing it a lot, *and* you don't court error by going through the process of copying it over again and again.