Exam 2 Review - Discrete Structures II - Fall 2020

Exam 2 will be on Thursday, November 5 from 6-8 pm. The exam will cover Chapter 11. Notes, calculator, and textbook are allowed, but no other resources (e.g. other people, the internet, etc.)

Outline of topics for exam 2:

Section 11.1: Operations

- Know basic terminology: Unary operation, binary operation
- Know the properties of operations: Commutative, associative, identity, inverse, idempotent, left / right distributive, distributive, involution, closure.
- Be able to identify and explain if an operation has a certain property or not.
- Understand and be able to construct operation tables.

Section 11.2: Algebraic Systems

- Know basic terminology: groups and basic notation for groups, abelian group.
- Given a set with a binary operation, determine if it's a group or not. If it is, prove it. If not, explain why not.

Section 11.3: Some General Properties of Groups

- Know the basic properties of identity and inverses (Theorems 11.3.1-9). Be able to prove the properties using the definition of group.
- Compute powers of elements of groups, and be able to use the basic properties given by Lemma 11.3.13 and Theorem 11.3.14.

Section 11.4: Greatest Common Divisors and the Integers Modulo n

- Do computations using modular arithmetic.
- $\mathbb{Z}_n$ is a group under addition.
  Be able to do computations with these groups, including the group operations, exponentiation, and finding inverses.

Section 11.5: Subsystems

- Use conditions (Theorem 11.5.3 / 11.5.5) for checking if a subset of a group is a subgroup.
- Given an element of a group, compute the cyclic group generated by that element.
- Determine if a subgroup is a cyclic group.
- Compute the order of an element in a group.

Section 11.6: Direct Products

- Do computations involving direct products of groups, including finding the identity element, inverses, and powers of elements.
- Be able to determine if a subset of a direct product is a subgroup.

Section 11.7: Isomorphisms

- Given two groups that are isomorphic, give an isomorphism and prove that it works using the definition.
- If two groups aren't ismorphic, explain why not.

Some Review Problems, with condensed answers

1. Suppose a binary operation $\diamond$ on the set $A = \{x, y, z\}$ is given by the table below:

$\diamond$	x	y	z
x	y	z	x
y	z	y	y
z	x	y	z

Which of the following properties does $\diamond$ have? Explain your answers.

(a) commutative (b) identity (c) inverse

(d) Is A a group under the binary operation of $\diamond$? Explain why or why not.

2. In each part, a group $[G; *]$ and two elements X, Y of G are given. Compute X^4, X^{-3}, and $(X * Y)^{-1}$. Also, state the identity for each group. (*Note:* Each group has a standard binary operation – use this for $*$ in each part.)

(a) Group: $\mathbb{Z}_6 \times \mathbb{Z}_8$
Elements: $X = (3, 5)$, $Y = (2, 7)$

(b) Group: $\mathbb{R}^* \times \mathbb{R}^2$
Elements $X = (3, 1, 4)$, $Y = (7, 2, 5)$.

3. For the group in part (a) of problem 2, compute the cyclic subgroup generated by X and the order of X.

4. For the group in part (b) of problem 2, compute five different elements in the cyclic group generated by X, and compute the order of X.

5. Let $S = \{1, 2, \ldots, 10\}$, and let $\mathcal{P} = \mathcal{P}(S)$ be the power set of S.

(a) Note that $\cup$ (i.e. set union) is a binary operation on $\mathcal{P}$. Is $\mathcal{P}$ a group under this binary operation? Explain which properties of the definition work and which properties fail. (You can assume the laws of set theory are true.)

(b) Note that set complement is a unary operation on $\mathcal{P}$. More precisely, under this operation, the set A is mapped to the set $A^c = S - A$. Is this unary operation an involution? Explain your answer.

6. (a) Prove that the set $H = \{(2x, 3^n) \,|\, x, n \in \mathbb{Z}\}$ is a subgroup of $\mathbb{R} \times \mathbb{R}^+$.

(b) Explain why the set $H = \{(2x, 3^n) \,|\, x, n \in \mathbb{Z}, n \geq 0\}$ is not a subgroup of $\mathbb{R} \times \mathbb{R}^+$.

7. Let $[G; *]$ be a group, where $G = \{u, v, x, y, z\}$ and suppose there's an isomorphism $f : \mathbb{Z}_5 \to G$ given by $f(0) = v$, $f(1) = y$, $f(2) = u$, $f(3) = x$, $f(4) = z$. Find the following. Express each answer as u, v, x, y or z.

(a) the identity of G. (b) the inverse of u (c) $u * z$ (d) x^3

8. Prove that $f : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = \ln x$ is an isomorphism using the definition.

9. A pair of groups is given in each part. Explain why the groups aren't isomorphic.

(a) $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$

(b) $\mathbb{Z}_6 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_8$

Answers:

1. $\diamond$ is commutative and has both the identity and inverse properties. A is not a group.
2. (a) $X^4 = (0, 4)$, $X^{-3} = (3, 1)$, $(X * Y)^{-1} = (1, 4)$.
The identity is $(0, 0)$.
(b) $X^4 = (81, 4, 16)$, $X^{-3} = (1/27, -3, -12)$,
$(X * Y)^{-1} = (1/21, -3, -9)$. The identity is $(1, 0, 0)$.
3. $\langle X \rangle = \{(0, 0), (0, 2), (0, 4), (0, 6), (3, 1), (3, 3), (3, 5), (3, 7)\}$
$\text{ord}(X) = 8$
4. Some elements in the subgroup $\langle X \rangle$ are $(1, 0, 0)$, $(3, 1, 4)$, $(9, 2, 8)$, $(27, 3, 12)$, $(81, 4, 16)$, $(1/3, -1, -4)$, $(1/9, -2, -8)$, $(1/27, -3, -12)$, $(1/81, -4, -16)$.
The element X has infinite order.
5. (a) Not a group (identity, inverse, not associative)
(b) Yes, the operation is an involution ($(A^c)^c = A$)
7. (a) v
(b) x
(c) y
(d) z