The Chinese Remainder Theorem

Let's understand Theorem 15.1.15 in the case $p = 2$. The Theorem says: Let n_1 and n_2 be integers that have no common factor greater than 1. Let $n = n_1 n_2$. Define

$$\theta : \mathbb{Z}_n \to \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$$

by

$$\theta(k) = (k_1, k_2)$$

where $0 \leq k_1 < n_1$, $0 \leq k_2 < n_2$, $k \equiv k_1 \pmod{n_1}$, and $k \equiv k_2 \pmod{n_2}$. (That is, k_1 is the remainder you get when you divide k by n_1, and k_2 is the remainder you get when you divide k by n_2.) Then θ is an isomorphism from $\mathbb{Z}_n$ into $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$.

Recall that an isomorphism between two groups $[G_1; *_1]$ and $[G_2; *_2]$ is a bijection f from G_1 to G_2 with the property that $f(a *_1 b) = f(a) *_2 f(b)$ for all a, b in G_1.

Let's understand the Chinese Remainder Theorem in the case where $n_1 = 2$ and $n_2 = 3$. $*_1$ is $+_6$ (mod 6 addition), the standard operation on $\mathbb{Z}_6$, and $*_2$ is $+_{2,3}$, the direct product operation on $\mathbb{Z}_2 \times \mathbb{Z}_3$ given by the formula $(x, y) +_{2,3} (x', y') = (x +_2 x', y +_3 y')$. Let θ be the function from $\mathbb{Z}_6$ to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_3$ given by the table below, showing k versus $\theta(k) = (k_1, k_2)$, where $0 \leq k_1 < 2$, $0 \leq k_2 < 3$, $k_1 \equiv k \pmod{2}$, and $k_2 \equiv k \pmod{3}$:

k	$\theta(k) = (k_1, k_2)$
0	$(0, 0)$
1	$(1, 1)$
2	$(0, 2)$
3	$(1, 0)$
4	$(0, 1)$
5	$(1, 2)$

Theorem 15.1.15 says that θ is an isomorphism from $\mathbb{Z}_6$ to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_3$. That is, $\theta(a +_6 b) = \theta(a) +_{2,3} \theta(b)$ for all a, b in $\mathbb{Z}_6$. Let's use the table to check this in one case (out of $6 \times 6 = 36$ cases). Does $\theta(3 +_6 5)$ equal $\theta(3) +_{2,3} \theta(5)$? I.e., does $\theta(2)$ equal $(1, 0) +_{2,3} (1, 2)$? I.e., does $(0, 2)$ equal $(1 +_2 1, 0 +_3 2)$? It does.

Also, the inverse function θ^{-1} is an isomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_3$ to $\mathbb{Z}_6$. That is, $\theta^{-1}((a, b) +_{2,3} (c, d)) = \theta^{-1}((a, b)) +_6 \theta^{-1}((c, d))$ for all $(a, b), (c, d)$

in $\mathbb{Z}_2 \times \mathbb{Z}_3$. Let's check this in one case. Does $\theta^{-1}((0,1) +_{2,3} (1,2))$ equal $\theta^{-1}((0,1)) +_6 \theta^{-1}((1,2))$? I.e., does $\theta^{-1}((1,0))$ equal $\theta^{-1}((0,1)) +_6 \theta^{-1}((1,2))$? I.e., does 3 equal $4 +_6 5$? It does.