Homomorphism of groups

Remember that an isomorphism from a group $[G_1, *_1]$ to a group $[G_2, *_2]$ is a bijection f from G_1 satisfying the relation

(*) $f(a *_1 b) = f(a) *_2 f(b)$ for all a, b in G_1.

If we *drop* the requirement that f be bijective, then what we have is the notion of a *homomorphism*. For instance, there is no bijection from $\mathbb{Z}$ to $\mathbb{Z}_2$, so there certainly isn't an isomorphism, but there is a map f from $\mathbb{Z}$ to $\mathbb{Z}_2$ that sends the even integers to 0 and the odd integers to 1, and it has property (*).

You can check that $f(a+b) = a +_2 b$ for all integers a, b. This means that if you want to know the remainder when you divide $a + b$ by 2, compute the remainder r that you get when you divide a by 2 and the remainder s that you get when you divide b by 2 and then compute $r +_2 s$.

Note that if f is a homomorphism, then the formula $f(a *_1 b) = f(a) *_2 f(b)$ extends automatically to bigger formulas like $f(a *_1 b *_1 c) = f(a) *_2 f(b) *_2 f(c)$.

There is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{10}$ that maps 17 to 7, 1023 to 3, -14 to 6, and more generally maps each nonnegative integer to its last digit (and maps each negative integer to 9 minus its last digit). That is, this homomorphism sends every n to $n\%10$. The homomorphism property is related to the fact that if you know the last digit of the positive integer a and the last digit of the positive integer b, then you can deduce the last digit of the positive integer $a + b$.

A historically important homomorphism is the function f from $\mathbb{Z}$ to $\mathbb{Z}_9$ that sends every n to $n\%9$. It is easy to compute $n\%9$ for any positive integer n: just add the digits of n, obtaining a new number n', and then add the digits of n', obtaining a new number n'', and so on, until you arrive at a single-digit number s; if s is 9, then $n\%9 = 0$, otherwise $n\%9 = s$. For instance, with $n = 1234567$, we get $n' = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ and $n'' = 2 + 8 = 10$ and $n''' = 1 + 0 = 1$, so $f(n) = n\%9 = 1$. This homomorphism is at the heart of the method of casting out nines, one of the earliest attempts at error-checking. The idea is that if you added two big integers a and b and got c, then $f(c)$ should be $f(a) +_9 f(b)$ in $\mathbb{Z}_9$. Turning this around, if $f(a) +_9 f(b)$ isn't $f(c)$, then we must have made a mistake when we computed c. This method doesn't tell you where you made the mistake – only that you made one. But for many purposes that's the right first step.

I'll conclude with two examples of homomorphisms related to Exercise

11.7.8(a) from the textbook. The operation abs($\cdot$) from $[\mathbb{R}^*, \times]$ to $[\mathbb{R}^+, \times]$ that sends x to $|x|$ is a homomorphism because $\text{abs}(\text{x} \times \text{y}) = \text{abs}(\text{x}) \times \text{abs}(\text{y})$ (that is, $|xy| = |x||y|$) for all nonzero x, y. Likewise, the operation sign($\cdot$) from $[\mathbb{R}^*, \times]$ to $[\{1, -1\}, \times]$ that sends x to $+1$ if x is positive and -1 if x is negative is a homomorphism because $\text{sign}(\text{xy}) = \text{sign}(\text{x})\text{sign}(\text{y})$ for all nonzero x, y.