The order sequence of a finite group

The “order sequence of a finite group” (not a standard term, but one we’ll use in this class) is the sequence whose terms are the respective orders of all the elements of the group, arranged in increasing order.

In $\mathbb{Z}_3$ the element 0 has order 1, the element 1 has order 3, and the element 2 has order 3, so the order sequence of this group is 1,3,3.

In $\mathbb{Z}_4$ the element 0 has order 1, the element 1 has order 4, the element 2 has order 2, and the element 3 has order 4, so the order sequence of this group is 1,2,4,4. (Note that I have arranged the numbers 1,4,2,4 in increasing order.)

Theorem: If G_1 and G_2 are finite groups and f is an isomorphism between them, with $g \in G_1$ and $f(g) \in G_2$, the order of g in G_1 equals the order of $f(g)$ in G_2.

Consequently:

Theorem: If two groups are isomorphic, they have the same order sequence.

The theorem is a handy tools for proving that two particular groups are not isomorphic. Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$; the element $(0, 0)$ has order 1 while the other elements $(0, 1)$, $(1, 0)$, and $(1, 1)$ each have order 2, implying that the order sequence is 1,2,2,2. Since this is different from the sequence 1,2,4,4, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to the group $\mathbb{Z}_4$.

Order sequences are also useful in helping one find isomorphisms. Consider the group $\mathbb{U}_5$ (the set $\{1, 2, 3, 4\}$ with mod-5 multiplication). Its order sequence is 1,2,4,4, which suggests that it might be isomorphic to $\mathbb{Z}_4$. In fact, any isomorphism f from $\mathbb{Z}_4$ to $\mathbb{U}_5$ must map 0 (the only element of order 1 in $\mathbb{Z}_4$) to 1 (the only element of order 1 in $\mathbb{U}_4$) and must map 2 (the only element of order 2 in $\mathbb{Z}_4$) to 4 (the only element of order 2 in $\mathbb{U}_4$). There are only two bijections f from $\mathbb{Z}_4$ to $\mathbb{U}_4$ satisfying $f(0) = 1$ and $f(2) = 4$, so these are the only two candidate isomorphisms (and both candidates turn out to be true isomorphisms).