Vectors and linear (in)dependence

**Vector space basics:** A key example of a vector space is $\mathbb{R}^2$, in which the vectors are written as ordered pairs $(x, y)$; the sum of two vectors is given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and the product of a scalar (i.e., number) and a vector is given by

$$c(x, y) = (cx, cy).$$

We often write $(1,0)$ as $i$ and $(0,1)$ as $j$, so that for instance $(3, 5) = (3, 0) + (0, 5) = 3(1, 0) + 5(0, 1) = 3i + 5j$. We represent $(x, y)$ by an arrow with its tail at the origin and its head at the point written as $(x, y)$ in Cartesian coordinates.

Vector spaces $\mathbb{R}^3$, $\mathbb{R}^4$, ... are defined similarly (even though they’re harder to picture). In $\mathbb{R}^3$, we write $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$. We have $(1, 2, 4) + (3, 2, 7) = (1 + 3, 2 + 2, 4 + 7) = (4, 4, 11)$ and $5(1, 2, 4) = (5 \times 1, 5 \times 2, 5 \times 4) = (5, 10, 20)$.

Question: Can $k$ be written as a linear combination of $i$ and $j$? That is, do there exist numbers $a, b$ such that

$$(0, 0, 1) = a(1, 0, 0) + b(0, 1, 0)?$$

That is, do there exist numbers $a, b$ such that $(0, 0, 1) = a(1, 0, 0) + b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0)$?

Answer: **No.** The vector equation $(0, 0, 1) = (a, b, 0)$ translates into the three-equation system $0 = a$, $0 = b$, $1 = 0$ which has no solution. So $k$ cannot be written as a linear combination of $i$ and $j$.

More generally, a vector space is a set that comes equipped with two binary operations, called vector addition and scalar multiplication; the sum of two vectors is a vector, and the product of a scalar and a vector is a vector. These operations are required to satisfy various axioms, such as

$$v + w = w + v$$

$$(u + v) + w = u + (v + w)$$

$$a(v + w) = av + aw$$

$$(a + b)v = av + bv$$

$$1v = v$$

$$a(bv) = (ab)v$$
There exists a vector $\mathbf{0}$ (the “zero vector”) such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$. For all $\mathbf{v}$ there exists $\mathbf{w}$ (written “$-\mathbf{v}$”) such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

For instance, let’s prove that for all $\mathbf{v}$, $0\mathbf{v} = \mathbf{0}$. We have $0 + 0\mathbf{v} = 0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$. Since $0 + 0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$, we can apply the cancellation rule (a consequence of the existence of additive inverses) to deduce $0 = 0\mathbf{v}$, as claimed.

We can also prove that $-\mathbf{v}$ (the additive inverse of $\mathbf{v}$) is equal to $(-1)\mathbf{v}$ (the product of the scalar $-1$ and the element $\mathbf{v}$). Note that

$$
\mathbf{v} + (-\mathbf{v}) = \mathbf{0} \text{ (by the definition of additive inverses)} \\
= 0\mathbf{v} \\
= (1 + (-1))\mathbf{v} \\
= 1\mathbf{v} + (-1)\mathbf{v} \\
= \mathbf{v} + (-1)\mathbf{v}
$$

so $\mathbf{v} + (-\mathbf{v}) = \mathbf{v} + (-1)\mathbf{v}$; applying the cancellation law, we get $-\mathbf{v} = (-1)\mathbf{v}$.

We define subtraction of vectors as follows:

$$
\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \mathbf{u} + (-1)\mathbf{v}
$$

**Linear dependence:** Linear dependence is easier to describe than its opposite (linear independence). Examples of linear dependence of vectors are

$$
\mathbf{u} + \mathbf{v} = \mathbf{w}
$$

and

$$
\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}.
$$

In the former case, we would say that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent; in the latter case, we would say that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly dependent. In each case, we can write one of the vectors as a linear combination of the others; for instance, in the case of $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$, we can write $\mathbf{u} = \mathbf{w} + \mathbf{x} - \mathbf{v}$ or $\mathbf{v} = \mathbf{w} + \mathbf{x} - \mathbf{u}$ or $\mathbf{w} = \mathbf{u} + \mathbf{v} - \mathbf{x}$ or $\mathbf{x} = \mathbf{u} + \mathbf{v} - \mathbf{w}$.

A uniform way to write linear dependencies is to move all the terms to one side of the equation, leaving $\mathbf{0}$ on the other side; for instance, $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$ becomes $\mathbf{u} + \mathbf{v} - \mathbf{w} - \mathbf{x} = \mathbf{0}$, or

$$
(1)\mathbf{u} + (1)\mathbf{v} + (-1)\mathbf{w} + (-1)\mathbf{x} = \mathbf{0}.
$$
More generally, any linear dependence between four vectors \( u, v, w, \) and \( x \) can be written in the form

\[
(a)u + (b)v + (c)w + (d)x = 0
\]

where the scalars \( a, b, c, \) and \( d \) are not all zero.

Example: If

\[
(2)u + (-3)v + (-4)w + (5)x = 0
\]

then each of the four vectors can be written as a linear combination of the other three: \( u = (3/2)v + (4/2)w - (5/2)x \), etc.

We say that a set of vectors \( \{ u, v, \ldots \} \) is linearly dependent if there exist coefficients \( a, b, \ldots \), not all equal to 0, such that \( au + bv + \ldots \) is the zero vector 0. Confusing special case: Under this definition, if \( v = 0 \), the set \( \{ v \} \) is linearly dependent; otherwise, the set \( \{ v \} \) is linearly independent.

We say that a set of vectors \( \{ u, v, \ldots \} \) is linearly independent if it is NOT linearly dependent. That is, a set of vectors \( \{ u, v, \ldots \} \) is linearly independent if the only choice of coefficients \( a, b, \ldots \) that make the equation \( au + bv + \ldots = 0 \) true is \( a = 0, b = 0, \ldots \)