Math 475, Problem Set #3: Solutions

A. *Section 3.6, problem 1. Also: How many of the four-digit numbers being considered satisfy (a) but not (b)? How many satisfy (b) but not (a)? How many satisfy neither (a) nor (b)?*

No constraints: $5 \times 5 \times 5 \times 5 = 625$ (by the multiplication principle).

(a): $5 \times 4 \times 3 \times 2 = 120$ (by the multiplication principle). Note that for this problem, the digits can be considered in any order.

(b): $5 \times 5 \times 5 \times 2 = 250$ (by the multiplication principle). Note that for this problem, the digits can be considered in any order.

(a) and (b): Looking at the digits *from right to left*, we get $2 \times 4 \times 3 \times 2 = 48$ (by the multiplication principle). Note that if we look at the digits from left to right, things don't work out so nicely!

(a) but not (b): $120 - 48 = 72$ (by the subtraction principle). Or, looking at the digits from right to left, we get $3 \times 4 \times 3 \times 2 = 72$ (by the multiplication principle). Once again, if we look at the digits from left to right, things don't work out so nicely.

(b) but not (a): $250 - 48 = 202$ (by the subtraction principle). (Incidentally, the fact that 202 is twice a three-digit prime, unlike 72 which is a product of single-digit primes, makes it unlikely that this problem can be solved by the multiplication principle. Some of my own research is concerned with counting problems for which the answers turn out to be highly composite numbers, like 72; I try to find proofs via the multiplication principle that explain why the answer has such small prime factors.)

Neither (a) nor (b): $625 - (48 + 72 + 202) = 625 - 322 = 303$ (by the subtraction principle), or $625 - 120 - 250 + 48 = 303$ (by the inclusion-exclusion principle).

B. *Section 3.6, problem 8. Note that two circular arrangements that differ by a rotation are to be regarded as the same, for purposes of counting.*

First, let's count the seating arrangements *without* regarding two solutions that differ by a rotation as the same solution. Then there are 12 choices for where the first gentleman goes. After that, there are only 5 choices for the second gentleman, 4 for the next, and so on. Then there are 6 choices for the first lady, 5 for the next, and so on. Hence the total number of possibilities is

$$12 \times 5 \times 4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1,$$

or $12 \times 5! \times 6!$ (which comes to 1036800).

Now let's return to the original problem, for which we assume we *don't* distinguish between two solutions that differ by a rotation. Then we must divide the preceding figure by 12, giving $5! \times 6! = 86400$.

A second solution: Call one of the gentlemen Al. Any of the 6 ladies can sit to Al's right. To her right, we can put any of the 5 remaining gentlemen. To his right, we can put any of the 5 remaining ladies. To her right, we can put any of the 4 remaining gentlemen. Etc. Hence the total number of different circular arrangements is $6 \times 5 \times 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 1 \times 1$, or $6!5!$.

A third solution: There are $6!/6 = 5!$ ways to seat the gentleman at a table for 6, and $6!/6 = 5!$ ways to seat the ladies at a table for 6. The number of ways to combine the two tables for 6 into one table for 12 is 6, because we have to decide which of the ladies sits to the right of Al and then everything else is forced by our choice of separate circular arrangements for the gentlemen and ladies. So the answer is $5! \times 5! \times 6$.

C. *How many genuinely different necklaces can be made by stringing together a red bead, an orange bead, a yellow bead, a green bead, a blue bead, and a violet bead? These beads are featureless, so the necklace has no discernible "front" or "back".*

If the necklace had a discernible front or back, this would be the same as counting the circular permutations of 6 distinct objects; the answer would be $P(6)/6 = 6!/6 = 5! = 120$. However, since this necklace has no front or back, counting in this way would be double-counting: for instance, the necklace with the beads occurring in the circular order ROYGBV is the same as the necklace with the beads occurring in the circular order RVBGYO, flipped back-to-front. So the list of 120 circular permutations contains each distinct necklace exactly twice, so the number of genuinely different necklaces is $120/2$, or 60.

Here's another way to see it: Hold the necklace up by the red bead. There are 5 possibilities for the bead clockwise from it, 4 possibilities for the bead clockwise from that, 3 possibilities for the next bead in clockwise order, and so on, so that the number of possibilities is 5x4x3x2x1, or 120. But this is double-counting, since we could flip the necklace back-to-front and it would still be the same necklace despite its different appearance. Since we've double-counted, the correct answer is 120/2, or 60.

D. *A woman invites a nonempty subset of twelve friends to a party.*

   (a) *In how many ways can she do it, if two of the friends are married to each other and must be invited together or not at all.*

   The married couple can be considered as one person, since they must be invited together or not at all. So we can think of the woman as inviting a nonempty subset of eleven friends to her party. We have two choices for each of the eleven people: invite that person or not. That gives us a contribution of 2^{11}. But then we've counted the empty subset of friends. So the number of ways is $2^{11} - 1$.

   (b) *Repeat part (a) if instead the two friends are recently divorced and cannot both be invited at the same time.*

   Now the number of ways is the total number of ways of inviting a nonempty subset of twelve friends ($2^{12} - 1$) minus the number of ways of inviting a nonempty subset of the twelve friends in which both of the divorced people are included (2^{10}). So the number of ways is $2^{12} - 1 - 2^{10} = 3071$, by the subtraction principle.

   Alternatively: There are $2^{10} - 1$ ways to invite neither member of the divorced couple (any non-empty subset of the other 10 will do), 2^{10} ways to invite the divorced husband and some subset of the non-divorced people, and 2^{10} ways to invite the divorced wife and some subset of the non-divorced people. So the number of ways is $(2^{10} - 1) + 2^{10} + 2^{10} = 3071$.

E. *You are dealt a hand of poker, that is, a set of 5 cards from the standard deck of 52 (the order of the cards within a hand does not matter). Assume that all hands are equally likely. Which is larger, the probability* *of getting a full house or the probability of getting four-of-a-kind? What is the ratio of the probabilities? To answer this, you must compute the number of different hands that contain a full house and the number of different hands that contain four-of-a-kind, see which is larger, and compute the ratio of these numbers. (Reminder: The deck contains 4 cards in each of 13 ranks. A full house is a hand that has three cards of one rank and two cards of a different rank. A four-of-a-kind is a hand that has four cards of one rank and one card of a different rank.)*

To choose a full house, you must choose a rank (which can be done in 13 ways) and then choose a different rank (which can be done in 12 ways). Then you must choose 3 cards from the first rank (which can be done in $C(4,3) = 4$ ways) and 2 cards from the second rank (which can be done in $C(4,2) = 6$ ways). Hence there are $13 \times 12 \times 4 \times 6$ distinct full houses, each with probability $1/C(52,5)$.

To choose four-of-a-kind, you must choose a rank (which can be done in 13 ways) and then choose a different rank (which can be done in 12 ways). Then you must choose 4 cards from the first rank (which can be done in $C(4,4) = 1$ way) and 1 cards from the second rank (which can be done in $C(4,1) = 4$ ways). Hence there are $13 \times 12 \times 1 \times 4$ distinct four-of-a-kind hands, each with probability $1/C(52,5)$.

Comparing, we see that a full house is exactly six times as likely as a four-of-a-kind.

Alternative solution: For those of you prefer permutations to combinations, we can count as follows: Each full house consists of three cards of one rank and two cards of another rank and so can be ordered in $3!2!$ ways so that the three cards of the same rank appear first and the two cards of the same rank appear last. The number of ways to have a 5-permutation of a deck of 52 cards such that the first three cards are one rank and the next two cards are of some other rank is $52 \times 3 \times 2 \times 48 \times 3$. Hence the number of different possible full-house hands is $(52)(3)(2)(48)(3)/(3!)(2!) = 3744$. Likewise, the number of different possible four-of-a-kind hands is $(52)(3)(2)(1)(48)/(4!)(1!) = 624$. The former is larger by a factor of 6.

F. *Twelve new students arrive at a certain wizarding academy, and must be sorted into four different houses, with three going to each house. In* *how many ways can this be done? Express your answer in terms of factorials, and cancel all factorials that occur in both the numerator and the denominator. (E.g., if you got the answer* $4!3!/3!2!$, *I would want you to write it as* $4!/2!$.*)*

For the sake of definiteness, let's call the houses Gryffindor, Hufflepuff, Ravenclaw and Slytherin. Three students can be assigned to Gryffindor in $\binom{12}{3} = 12!/3!9!$ ways. Three students from the remaining 9 can be assigned to Hufflepuff in $\binom{9}{3} = 9!/3!6!$ ways. Three students from the remaining 6 can be assigned to Ravenclaw in $\binom{6}{3} = 6!/3!3!$ ways. The remaining 3 students must be assigned to Slytherin. The total number of possible assignments is therefore

$$\frac{12!}{3!9!} \frac{9!}{3!6!} \frac{6!}{3!3!};$$

when the common factors of 9! and 6! are cancelled, we are left with

$$\frac{12!}{3!3!3!3!}.$$

Note that this is just like assigning objects to boxes where the boxes are labeled (see Theorem 3.4.3).