Solutions to Final Exam

1. Three married couples are seated together at the counter at Monty's Blue Plate Diner, occupying six consecutive seats. How many arrangements are there with no wife sitting next to her own husband? We do *not* require men and women to alternate. For full credit, you must use the inclusion-exclusion method.

Solution: If we ignore the constraint about who can sit next to whom, the number of arrangements of the six people is $6!$. Let A_i be the set of arrangements with couple i sitting together ($i = 1, 2, 3$). $|A_i| = 2 \cdot 5!$: there are 2 ways to order couple i (man on the right or woman on the right) and then there are $5!$ ways to order couple i with the other 4 individuals (think of couple i as one entity and the remaining 4 people as individuals). Likewise $|A_i \cap A_j| = 2 \cdot 2 \cdot 4!$ and $|A_i \cap A_j \cap A_k| = 2 \cdot 2 \cdot 2 \cdot 3!$. Hence, using inclusion-exclusion, we see that the number of ways to arrange three married couples so that no wife sits next to her own husband is $6! - 3 \cdot (2 \cdot 5!) + 3 \cdot (2^2 \cdot 4!) - 2^3 \cdot 3! = 720 - 720 + 288 - 48 = 240$.

2. Use the method of characteristic equations to solve the linear recurrence $h_n = 4h_{n-1} - 4h_{n-2}$ with initial conditions $h_0 = 1$, $h_1 = 4$.

Solution: The characteristic equation is $x^2 = 4x - 4$, or $x^2 - 4x + 4 = 0$, which has 2 as a double root. Hence the solution can be expressed in the form $h_n = A2^n + Bn2^n$. Thus $1 = h_0 = (A)(2^0) + (B)(0)(2^0) = A$ and $4 = h_1 = (A)(2^1) + (B)(1)(2^1) = 2A + 2B$, which yields $A = 1$ and $B = 1$. Therefore $h_n = 2^n + n2^n = (n+1)2^n$.

3. Solve the non-homogeneous linear recurrrence $h_n = 3h_{n-1} + 1$ with initial condition $h_0 = 0$.

Solution: The general solution to the corresponding homogeneous recurrence $h'_n = 3h'_{n-1}$ is $h'_n = A3^n$. To find a particular solution to the non-homogeneous recurrence, we can guess an answer of the same general form as the non-homogeneous term 1; that is, we guess $h_n = B$ for some constant B. Plugging this into $h_n = 3h_{n-1} + 1$ we get $B = 3B + 1$ which requires $B = -\frac{1}{2}$. This corresponds to the particular solution $h_n = -\frac{1}{2}$. So the solution we seek is of the form $h_n = h'_n - \frac{1}{2} = A3^n - \frac{1}{2}$. Since $0 = h_0 = A3^0 - \frac{1}{2} = A - \frac{1}{2}$, we get $A = \frac{1}{2}$. Hence $h_n = \frac{1}{2} \cdot 3^n - \frac{1}{2}$.

Alternative solution: The non-homogeneous term 1 satisfies a linear recurrence with characteristic polynomial $(x-1)$. Since the homogeneous version of the original recurrence has characteristic polynomial $x-3$, we can multiply the two and conclude that the original sequence h_n satisfies a linear recurrence with characteristic polynomial $(x-1)(x-3)$. It follows that we can write $h_n = A + B3^n$ for suitable constants A, B. Putting $0 = h_0 = A + B$ and $1 = h_1 = A + 3B$ (where h_1 is computed via the original non-homogeneous recurrence) and solving this linear system, we get $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Hence $h_n = -\frac{1}{2} + \frac{1}{2} \cdot 3^n$.

4. For $n > 0$, let a_n be the number of ways to tile a 1-by-n strip with 1-by-2 tiles and 1-by-3 tiles (so that $a_0 = 1$, $a_1 = 0$, and $a_2 = 1$). Find a third-order recurrence relation satisfied by a_n, and write the generating function $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ as a rational function of x. **Do not solve for a_n.**

Solution: For $n \geq 3$, a tiling of the 1-by-n strip can begin with either a 1-by-2 tile (in which case the number of ways to complete the tiling is a_{n-2}) or a 1-by-3 tile (in which case the number of ways to complete the tiling is a_{n-3}). Hence $a_n = a_{n-2} + a_{n-3}$ for all $n \geq 3$, which is a recurrence with characteristic polynomial $x^3 - x - 1$. It follows that $a_0 + a_1 x + a_2 x^2 + \ldots$ can be written in the form $(A + Bx + Cx^2)/(1 - x^2 - x^3)$, where we get the denominator by taking the coefficients of $x^3 - x - 1$ and applying them to the powers of x in increasing rather than decreasing order. There are two ways to solve for A, B, C:

Method 1: Multiply both sides of

$$(A + Bx + Cx^2)/(1 - x^2 - x^3) = a_0 + a_1 x + a_2 x^2 + \ldots$$

by $1 - x^2 - x^3$, obtaining

$$A + Bx + Cx^2 = (1 - x^2 - x^3)(a_0 + a_1 x + a_2 x^2 + \ldots).$$

Equating coefficients of successive powers of x, we get $A = a_0 = 1$, $B = a_1 = 0$, and $C = a_2 - a_0 = 0$, so the generating function is $1/(1 - x^2 - x^3)$.

Method 2: Multiply both sides of the recurrence $a_n = a_{n-2} + a_{n-3}$ by x^n and sum over all $n \geq 3$. This yields

$$f(x) - a_0 - a_1 x - a_2 x^2 = x^2[f(x) - a_0] + x^3[f(x)].$$

Plugging in $a_0 = 1$, $a_1 = 0$, and $a_2 = 1$, we get

$$f(x) - 1 - x^2 = x^2[f(x) - 1] + x^3[f(x)],$$

which becomes $(1 - x^2 - x^3)f(x) = 1$, so $f(x) = 1/(1 - x^2 - x^3)$.

5. For $n > 0$, let a_n be the number of length-n strings of 1's, 2's, 3's and 4's in which no two 4's appear consecutively. Find a second-order recurrence relation satisfied by a_n. **Do not solve for a_n.**

Solution: An allowed string of length n can begin with a 1, in which case there are a_{n-1} ways to finish it; or it can begin with a 2, in which case there are a_{n-1} ways to finish it; or it can begin with a 3, in which case there are a_{n-1} ways to finish it; or it can begin with a 4, in which case the next symbol must be a 1, 2, or 3 (3 choices), and then there are a_{n-2} ways to finish the string, giving $3a_{n-2}$ strings of length n that start with a 4. Hence the total number of possibilities satisfies $a_n = a_{n-1} + a_{n-1} + a_{n-1} + 3a_{n-2} = 3a_{n-1} + 3a_{n-2}$.

6. (a) Find the polynomial $p(n)$ of degree 2 whose difference table is

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1   3  11  25  45 ...
  2   8  14  20 ...
    6   6   6 ...
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(where the leftmost entry in the top row is $p(0)$). Express your answer in the form $An^2 + Bn + C$, and check it for $n = 0$, 1, 2, and 3.

(b) Find a formula for the sum $\sum_{k=0}^{n} p(k)$. Express your final answer in the form $An^3 + Bn^2 + Cn + D$, and check it for $n = 0$, 1, 2, and 3.

Solution:

(a) Reading down the left diagonal, we see that $p(n)$ can be written as $1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 6 \cdot \binom{n}{2} = 1 + 2n + 3n(n-1) = 3n^2 - n + 1$.

(b) Using those same coefficients, we get $1 \cdot \binom{n+1}{1} + 2 \cdot \binom{n+1}{2} + 6 \cdot \binom{n+1}{3}$. This becomes $(n+1) + (n+1)n + (n+1)n(n-1) = n^3 + n^2 + n + 1$.

7. Let h_n be the number of length-n strings of 1's, 2's, and 3's in which 1's occur an even number of times and 2's occur an even number of times. Find an exact formula for the exponential generating function of the sequence $h_0, h_1, h_2, \ldots$, and use it to find an exact formula for h_n (valid for all $n \geq 0$).

Solution: The exponential generating function for strings consisting of an even number of 1's (and nothing else) is $(e^x + e^{-x})/2$. The exponential generating function for strings consisting of an even number of 2's (and nothing else) is $(e^x + e^{-x})/2$. The exponential generating function for strings consisting of an arbitrary number of 3's is e^x. Hence the exponential generating function for strings consisting of an even number 1's, an even number of 2's, and an arbitrary numbers of 3's is $(e^x+e^{-x})/2 \cdot (e^x+e^{-x})/2 \cdot e^x$, or $\frac{1}{4}(e^{3x}+2e^x+e^{-x})$. The coefficient of $x^n/n!$ in this expansion is $\frac{1}{4}(3^n + 2 \cdot 1^n + (-1)^n)$, so the number of such strings is is $\frac{1}{4}(3^n + 2 + (-1)^n)$.

8. Express $S(n, n-1)$ (for $n \geq 1$) as a polynomial in n, and prove that your formula is valid by using the combinatorial interpretation of Stirling numbers of the second kind.

Solution: $S(n, n-1)$ is the number of ways to divide n objects into $n-1$ disjoint subsets. Exactly one of these subsets must be of size 2, with the rest being of size 1. Hence, such a division of the n objects is equivalent to choosing 2 of the n objects to form a set of size 2 (and letting the other $n-2$ elements form sets of size 1 by themselves). Therefore $S(n, n-1) = \binom{n}{2} = n(n-1)/2$.

9. A baton is divided into $2n + 1$ cylindrical bands of equal length ($n \geq 1$). In how many different ways can the $2n + 1$ bands be colored if 3 colors are available, assuming that no two adjacent bands may be given the same color? (Two colorings count as the same if one of them can be converted into the other by turning the baton around.)

Solution: Here the group acting on the set of colorings is just the 2-element group consisting of the identity element (don't turn the baton) and a non-identity element (do turn the baton). By Burnside's Lemma, the answer can be written as $\frac{1}{2}(A + B)$, where A is the number of of allowed colorings that are fixed by the identity operation (that is, the total number of allowed colorings), and B is the number of of allowed colorings that are fixed by the turn operation. A equals $3 \cdot 2^{2n}$, since (starting at one end of the baton) there are 3 possible colors to use at the end, and at each subsequent band there are exactly 2 possible colors to use (namely, the two colors that differ from the one just used). On the other hand, B equals $3 \cdot 2^n$, since (starting from the middle of the baton) there are 3 possible colors to use, and at each subsequent band there are exactly 2 possible colors to use (where you must make sure to use the same color for each band and for its mate on the other side of the baton). So the answer is $\frac{3}{2}(4^n + 2^n)$.

10. Use Burnside's Lemma to count the number of circular 6-permutations of the multiset $\{2 \cdot \mathrm{R},\ 2 \cdot \mathrm{W},\ 2 \cdot \mathrm{B}\}$.

Solution: Here the group acting on the set of colorings is the 6-element group consisting of rotations by multiple of 60 degrees. By Burnside's Lemma, the answer is $\frac{1}{6}(A + 2B + 2C + D)$, where A is the number of colorings that are invariant under 0 degree rotation, B is both the number of colorings that are invariant under 60 degree clockwise rotation and the number of colorings that are invariant under 60 degree counterclockwise rotation, C is both the number of colorings that are invariant under 120 degree clockwise rotation and the number of colorings that are invariant under 120 degree counterclockwise rotation, and D is the number of colorings that are invariant under 180 degree rotation. We have $A = 6!/2!2!2! = 90$ (these are just ordinary permutations of the multiset), $B = 0$ (the only way a circular 6-permutation involving R's, W's, and B's can be invariant under 60 degree rotation is if all six letters are the same), $C = 0$ (for the same sort of reason as $B = 0$, but more complicated: a circular 6-permutation involving R's, W's, and B's can be invariant under 120 degree rotation only if either some color occurs six times or two of the colors occur three times each, contradicting our requirement that each of the three colors occurs twice), and $D = 3! = 6$ (there are 3! ways of assigning the three colors to the three pairs of diametrically opposite positions in the circular 6-permutation, and each of these corresponds to a way of arranging two R's two B's, and two W's). So the number of orbits, i.e., the number of 6-permutations, is $\frac{1}{6}(90 + 6) = 16$.