1. Let \( p(n) \) be the number of unconstrained partitions of \( n \) if \( n \geq 0 \), and 0 otherwise, so that

\[
p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \ldots
\]

for all \( n > 0 \). Use the recurrence for \( p(n) \) to compute the last digit of \( p(n) \) for every \( n \) between 1 and 1000. Make a conjecture about the relationship between the last digit of \( n \) and the last digit of \( p(n) \); specifically, make a conjecture about which pairs \( (n \mod 10, \ p(n) \mod 10) \) occur and which don’t.

Here’s a Maple program that does this:

```maple
F := proc(n) option remember; local total, k;
if n=0 then 1; elif n<0 then 0; else total := 0;
k := 1; while k*(3*k+1)/2 <= n do
  total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k+1: od:
k := -1; while k*(3*k+1)/2 <= n do
  total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k-1: od:
total mod 10; fi: end;
```

We then create a matrix to keep track of how often it happens that \( n \) ends with the digit \( i \) while \( p(n) \) ends with the digit \( j \) (for \( i, j \) between 0 and 9), and print out its entries:

```maple
for i from 0 to 9 do for j from 0 to 9 do a[i,j]:=0: od: od:
for n from 1 to 1000 do k := F(n);
a[n mod 10, k mod 10] := a[n mod 10, k mod 10] + 1; od:
for i from 0 to 9 do seq(a[i,j],j=0..9) od;
```

This results in the output

\[
\begin{array}{cccccccc}
14, & 7, & 13, & 12, & 3, & 5, & 12, & 15, & 9, & 10 \\
9, & 11, & 14, & 9, & 8, & 9, & 10, & 13, & 7, & 10 \\
3, & 14, & 12, & 14, & 10, & 8, & 8, & 12, & 6, & 13 \\
8, & 9, & 12, & 9, & 8, & 17, & 5, & 13, & 9, & 10 \\
\end{array}
\]
from which we conjecture that when \( n \) ends in 4 or 9, \( p(n) \) ends in 0 or 5. That is, if \( n \) is 1 less than a multiple of 5, \( p(n) \) is a multiple of 5. This fact was first noticed and proved by Ramanujan. Coincidentally, Prof. Ono spoke about this very result in his Math Club talk yesterday (December 1)!

2. Let \( f(0) = 1 \) and recursively define \( f(n) = f(n-1) + f(n-3) - f(n-6) - f(n-10) + f(n-15) + f(n-21) + \cdots \) for all \( n > 0 \), where terms of the form \( f(n-k) \) are to be ignored once \( k > n \).

(a) Since the formal power series \( F(q) = \sum_{n \geq 0} f(n) q^n = 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 7q^7 + \cdots \) has constant term 1, we saw in class that it admits a (unique) convergent infinite formal product expansion of the form

\[
(1 - q)^{a_1}(1 - q^2)^{a_2}(1 - q^3)^{a_3}(1 - q^4)^{a_4} \cdots
\]

Find \( a_1 \) through \( a_{24} \), and conjecture a general rule.

The defining recursion for \( f(n) \) tells us that \( F(q) = 1/(1-q-q^3+q^6+q^{10}-q^{15}-q^{21}+\ldots) \). So we may start from this expression (say using exponents up to 25), take its formal Taylor series, and repeatedly divide or multiply by a suitable expression of the form \((1 - q^m)^k \) so as to increase the degree of the first non-constant term in the (continually modified) series.

More specifically, suppose we start with the command

\[
taylor(1/(1-q-q^\_3+q^\_6+q^\_10-q^\_15-q^\_21),q,25);
\]

which returns a power series with constant term 1 and linear term \( q \). We can get the coefficient of the linear term to become 0 if we multiply by \( 1 - q \). Entering

\[
taylor(%*(1-q),q,25);
\]
we get a power series with constant term 1, no linear term, no quadratic term, and a cubic term \( q^3 \). We can get the coefficient of the cubic term to become 0 if we multiply by \( 1 - q^3 \). So we enter

\[
taylor(\%*(1-q^3),q,25);
\]

And so on. Proceeding in this fashion, we find that

\[
a_1 = -1, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = -1, \quad a_5 = -1, \quad a_6 = 0, \quad a_7 = -1, \quad a_8 = -1, \quad \text{etc.}
\]

that is, for all \( n \leq 24 \), \( a_n \) is 0 if \( n \) is 2 more than a multiple of 4 and is \(-1\) otherwise.

(b) Assuming that your answer from (a) is correct, prove that for a particular set \( S \) of positive integers (which you must find!), \( f(n) \) equals the number of partitions of \( n \) into parts belonging to \( S \).

Since \( a_n \) is always either \(-1\) or \(0\), this is simple: \((1 - q)^a_1(1 - q^2)^a_2(1 - q^3)^a_3(1 - q^4)^a_4 \cdots \) is \( \prod_{k \in S} (1 - q^k)^{-1} \), so \( S \) is the set of all positive integers that are either odd or divisible by 4.

(c) Prove that your conjectures from (a) and (b) are correct, e.g. by using the Jacobi triple product identity

\[
\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1}z^2)(1 + x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}
\]

(which you do not need to prove). An equivalent form of the Jacobi triple product identity is

\[
\prod_{i=1}^{\infty} (1 + xq^i)(1 + x^{-1}q^{-i-1})(1 - q^i) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} x^n.
\]

We want to equate the sum

\[
\ldots + x^9z^6 + x^4z^4 + x^1z^2 + x^0z^0 + x^1z^2 + x^4z^4 + x^9z^6 + \ldots
\]

(the right hand side of the Jacobi triple product identity) with the sum

\[
\ldots - q^{15} + q^6 - q + 1 - q^3 + q^{10} - q^{21} + \ldots
\]

(the \( q \)-series we are trying to evaluate in product form) by making a suitable choice for \( x \) and \( z \); we achieve this by setting \( x = q^2 \) and \( z = i\sqrt{q} \) (so that \( z^2 = -q \)). Then the left hand side of the Jacobi
triple product identity becomes $\prod_{i=1}^{\infty} (1 - q^{4n^2})(1 - q^{4n-1})(1 - q^{4n-3})$.
So we have shown that

$$1/F(q) = \prod \frac{1}{1 - q^m}$$

where the product is taken over all $m$ that are not 2 more than a multiple of 4. This is equivalent to what was conjectured in part (b).