1. There is a unique polynomial of degree $d$ such that $f(k) = 2^k$ for $k = 0, 1, \ldots, d$. What is $f(d + 1)$? What is $f(-1)$?

Suppose $g(k)$ is a polynomial of degree $m \geq 1$, so that its sequence of $m$th differences is constant. If we define $G(k) = g(k) + g(k - 1) + \ldots + g(1)$ for all $k \geq 1$, then the first differences of $G$ are the “zeroth” differences of $g$, the second differences of $G$ are the first differences of $g$, and so on, so that the sequence of $m + 1$st difference of $G$ is constant, implying that $G(k)$ is given by a polynomial of degree $m + 1$ in $k$. This last assertion is true for $g(k - 1) + g(k - 2) + \ldots + g(0) + 1$ as well, since it differs from $G(k)$ by the substitution of $k - 1$ for $k$ and the addition of the constant 1.

In particular, we see that if $f$ is a polynomial of degree $d - 1$ with $f(k) = 2^k$ for $0 \leq k \leq d - 1$, then the sum $F(k) = f(k - 1) + f(k - 2) + \ldots + f(0) + 1$ defines a polynomial function of degree $d$, and it is easy to see that if $f$ satisfies the property that characterizes $f_{d-1}$, $F$ satisfies the property that characterizes $f_d$. Hence we have

$$f_d(k) = f_{d-1}(k - 1) + f_{d-1}(k - 2) + \ldots + f_{d-1}(0) + 1$$

for all $k \geq 0$ (not just $0 \leq k \leq d$), with the proviso that in the case $k = 0$, the only term on the right hand side is the 1.

Putting $k = d + 1$, we have $f_d(d + 1) = f_{d-1}(d) + f_{d-1}(d - 1) + \ldots + f_{d-1}(0) + 1 = f_{d-1}(d) + 2^{d-1} + \ldots + 1 + 1 = f_{d-1}(d) + 2^d$. That is, the sequence $f_0(1), f_1(2), f_2(3), \ldots$, has the sequence 1, 2, 4, \ldots as its sequence of first differences, from which it follows (say by induction) that $f_{d-1}(d) = 2^d - 1$.

On the other hand, for each fixed $d$ the relation $f_d(k) - f_d(k - 1) = f_{d-1}(k - 1)$ holds for all $k$, since it holds for all positive $k$ and since both sides of the equation are polynomials. Hence we have $f_d(0) - f_d(-1) = f_{d-1}(-1)$. Rewriting this as $f_d(-1) = f_d(0) - f_{d-1}(-1)$ and using the fact that $f_d(0) = 1$, we have $f_d(-1) = 1 - f_{d-1}(-1)$, from which it follows (say by induction) that $f_d(-1) = 1$ when $d$ is even and 0 when $d$ is odd. (Or, if you prefer formulas, $f_d = (1 + (-1)^n)/2$.}

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Note that you don’t need to have an explicit formula for $f_d(k)$ in order to solve this problem!