Consider the subset of the square grid bounded by the vertices \((0,0), (m,0), (0,n),\) and \((m,n)\), and let \(q\) be a formal indeterminate. Let the weight of the horizontal grid-edge joining \((i,j)\) and \((i+1,j)\) be \(q^j\) (for all \(0 \leq i \leq m-1\) and \(0 \leq j \leq n\)), and let each vertical grid-edge have weight 1. Define the weight of a lattice path of length \(m+n\) from \((0,0)\) to \((m,n)\) to be the product of the weights of all its constituent edges. Let \(P(m,n)\) be the sum of the weights of all the lattice paths of length \(m+n\) from \((0,0)\) to \((m,n)\), a polynomial in \(q\). (Note that putting \(q = 1\) turns \(P(m,n)\) into the number of lattice paths of length \(m+n\) from \((0,0)\) to \((m,n)\), which is the binomial coefficient \(\binom{m+n}{m!}n!\).)

(a) Give a formula for \(P(1,n)\) and for the generating function \[\sum_{n \geq 0} P(1,n)x^n.\]

(b) Find (and justify) a recurrence relation relating the polynomials \(P(m,n)\), \(P(m-1,n)\), and \(P(m,n-1)\) that generalizes the Pascal triangle relation for binomial coefficients.

(c) Let \(F_m(x)\) denote \(\sum_{n \geq 0} P(m,n)x^n\). Use your answer from (b) to give a formula for \(F_m(x)\) in terms of \(F_{m-1}(x)\), and from this derive a non-recursive formula for \(F_m(x)\).

(d) Write a computer program to compute the polynomial \(P(m,n)\) for any input values \(m, n\).

(e) Compute \(P(m,n)/P(m-1,n)\) for various values of \(m \geq 1\) and \(n \geq 0\) and conjecture a formula for it. Do the same for the ratio \(P(m,n)/P(m,n-1)\) with \(m \geq 0\) and \(n \geq 1\).

(f) Use the recurrence relation from (b) to verify your conjectures from (e).