Math 491, Problem Set #14
(due 11/13/03)

Consider the subset of the square grid bounded by the vertices $(0,0)$, $(m,0)$, $(0,n)$, and (m,n), and let q be a formal indeterminate. Let the weight of the horizontal grid-edge joining (i,j) and $(i+1,j)$ be q^j (for all $0 \le i \le m-1$ and $0 \le j \le n$), and let each vertical grid-edge have weight 1. Define the weight of a lattice path of length $m+n$ from $(0,0)$ to (m,n) to be the product of the weights of all its constituent edges. Let $P(m,n)$ be the sum of the weights of all the lattice paths of length $m+n$ from $(0,0)$ to (m,n), a polynomial in q. (Note that putting $q=1$ turns $P(m,n)$ into the number of lattice paths of length $m+n$ from $(0,0)$ to (m,n), which is the binomial coefficient $\frac{(m+n)!}{m!n!}$.)

(a) Give a formula for $P(1,n)$ and for the generating function

$$\sum_{n\geq 0} P(1,n)x^n.$$

(b) Find (and justify) a recurrence relation relating the polynomials $P(m,n)$, $P(m-1,n)$, and $P(m,n-1)$ that generalizes the Pascal triangle relation for binomial coefficients.

(c) Let $F_m(x)$ denote $\sum_{n\geq 0} P(m,n)x^n$. Use your answer from (b) to give a formula for $F_m(x)$ in terms of $F_{m-1}(x)$, and from this derive a non-recursive formula for $F_m(x)$.

(d) Write a computer program to compute the polynomial $P(m,n)$ for any input values m,n.

(e) Compute $P(m,n)/P(m-1,n)$ for various values of $m \ge 1$ and $n \ge 0$ and conjecture a formula for it. Do the same for the ratio $P(m,n)/P(m,n-1)$ with $m \ge 0$ and $n \ge 1$.

(f) Use the recurrence relation from (b) to verify your conjectures from (e).