Math 491, Problem Set #19
(due 12/9/03)

1. Let $P(n)$ and $Q(n)$ denote the numerator and denominator obtained when the continued fraction

$$x_1 + (y_1/(x_2 + (y_2/(x_3 + (y_3/\cdots + (y_{n-2}/(x_{n-1} + (y_{n-1}/x_n))\cdots))))))$$

is expressed as an ordinary fraction. Thus $P(n)$ and $Q(n)$ are polynomials in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_{n-1}$.

(a) By examining small cases, give a conjectural bijection between the terms of the polynomial $P(n)$ and domino tilings of the 2-by-$n$ rectangle, and a similar bijection between the terms of the polynomial $Q(n)$ and domino tilings of the 2-by-$(n-1)$ rectangle, as well as a conjecture that gives all the coefficients.

(b) Prove your conjectures from part (a) by induction on $n$.

2. Let $R(n)$ denote the determinant of the $n$-by-$n$ matrix $M$ whose $i,j$th entry is equal to

$$\begin{cases} x_i & \text{if } j = i, \\ y_i & \text{if } j = i + 1, \\ z_{i-1} & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) By examining small cases, give a conjectural bijection between the terms of the polynomial $R(n)$ and domino tilings of the 2-by-$n$ rectangle, and a conjecture for the coefficients.

(b) Prove your conjectures from part (a) by induction on $n$. 