1. (from unpublished work of Douglas Zare) Let $G_{m,n}$ be the directed graph with vertex set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq i \leq m, 0 \leq j \leq n\}$, with an arc from $(i,j)$ to $(i',j')$ iff $(j' - j, i' - i)$ is $(1,0)$, $(0,1)$, or $(1,1)$.

(a) For any legal path $P$ in $G_{m,n}$ from $(0,0)$ to $(m,n)$, define $d(P)$ as the number of diagonal steps in $P$ plus the number of upward steps in $P$ that are followed immediately by a rightward step. Show that the number of paths $P$ with $d(P) = k$ is exactly $2^k \binom{m}{k} \binom{n}{k}$.

(b) Let $M$ be the $(n+1)$-by-$(n+1)$ matrix with rows and columns indexed from 0 through $n$ whose $i,j$th entry is the total number of paths in $G_{i,j}$ from $(0,0)$ to $(i,j)$. Use the result of part (a) to find the LDU decomposition of $M$. That is: find square matrices $L$, $D$, $U$ such that $LDU = M$, where $L$ (resp. $U$) is a lower (resp. upper) triangular matrix with 1’s on the diagonal and where $D$ is a diagonal matrix (whose diagonal entries are permitted to be different). Use this in turn to compute det$(M)$.

(c) Interpret $M$ as the Lindstrom matrix of some directed graph and use this in turn to interpret det$(M)$ as the number of perfect matchings of some graph $H_n$. Be explicit about what $H_n$ looks like.