Reading and homework assignment

For 9/27, read 11.1 - 11.2 in Grinstead and Snell.

For 10/4, read 11.3 - 11.5 in Grinstead and Snell AND do assignment #2: http://jamesprop-p.org/584/P2.pdf

***Mathematica* reimbursement**

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Random variables (mostly review)

Probability spaces

A probability space is a set Ω equipped with a probability measure

$m(.)$, i.e., a real-valued function $m(.)$ satisfying two properties:
(1) $m(\omega) \geq 0$ for all ω in Ω, and
(2) $\sum_{\omega \text{ in } \Omega} m(\omega) = 1$.
(Interpretation: Ω is a set of all possible outcomes of a one-step process, like the roll of a die, or the set of all possible histories of a many-step process, like tossing a coin repeatedly; for each ω in Ω, $m(\omega)$ is the probability of the outcome/history ω.)

For any subset E of Ω, we define P(E) (or sometimes "Prob(E)") as

$$\sum_{\omega \text{ in } E} m(\omega).$$

We call E an <u>event</u>, and we call P(E) the <u>probability</u> of E.

We say two events E and F are <u>independent</u> if P($E \cap F$), also written P(E and F), is equal to
P(E) P(F).

Example: Let $\Omega = \{1,2,3,4,5,6\}$ with $m(\omega) = \frac{1}{6}$ for all ω in Ω, corresponding to the rolls of a fair die. The event "We roll an odd number" is the set E = {1,3,5}, the event "We roll a square" is the event F = {1,4}, and the event "We roll an odd number that is also a perfect square" is the event $E \cap F$ = {1}. Since

$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$,

$P(F) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$, and

$P(E \cap F) = \frac{1}{6} = (\frac{1}{2})(\frac{1}{3}) = P(E)\ P(F)$,

the events E and F are independent.

Note: $P(\{\omega\}) = m(\omega)$.

```
(* This is a Mathematica comment.*)

(* Does Mathematica let us define a function by example rather than by a general rule? *)

In[2]:= For[ω = 1, ω ≤ 6, ω++, m[ω] = 1 / 6]

In[3]:= m[7]

Out[3]= m[7]

In[4]:= m[3]

Out[4]= 1/6

(* Yes it does! *)

In[5]:= Prob[E_] := Sum[m[E[[i]]], {i, Length[E]}]

In[6]:= Prob[{1, 3, 5}]

Out[6]= 1/2

In[7]:= Prob[{1, 4}]

Out[7]= 1/3

In[8]:= Prob[Intersection[{1, 3, 5}, {1, 4}]]

Out[8]= 1/6

In[9]:= Indep[E_, F_] := Prob[Intersection[E, F]] == Prob[E] Prob[F]

In[10]:= Indep[{1, 3, 5}, {1, 4}]

Out[10]= True

In[11]:= Indep[{1, 3, 5}, {1, 5}]

Out[11]= False
```

Random variables as functions

A random variable (or "r.v.") is a real-valued function on a probability space.

Let X be a real-valued function on Ω, and let a be some real number. We define the event $X=a$ as the set $\{\omega: X(\omega)=a\}$.

Example: $\Omega = \{HH,HT,TH,TT\}$, with $m(\omega)=\frac{1}{4}$ for all ω.

Let

$$Z(\mathrm{HH})=0,$$

$Z(HT)=Z(TH)=1$, and
$Z(TT)=2$.

That is, the random variable Z is the number of Tails that turn up in two tosses of a fair coin. We have

$P(Z=0) = P(\{HH\}) = \frac{1}{4}$,

$P(Z=1) = P(\{HT,TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and

$P(Z=2) = P(\{TT\}) = \frac{1}{4}$.

We recognize this as the probability distribution function of a binomial random variable with parameters

n=2 and p=$\frac{1}{2}$.

Consider the random variable Z' given by

$$Z'(\text{HH})=2,$$
$$Z'(\text{HT})=Z'(\text{TH})=1, \text{ and}$$
$$Z'(\text{TT})=0.$$

That is, the random variable Z' is the number of Heads that turn up in those same two tosses of the coin. Z', like Z, is a binomial random variable with parameters n=2 and $p=\frac{1}{2}$. They have the same probability distribution or probability law. But Z' is not the same random variable as Z, because it is not the same function on the set Ω = {HH,HT,TH,TT}.

A random variable taking only the values 0 and 1 is called a <u>Bernoulli</u> r.v. We say X is Bernoulli with parameter p if P(X=1) = p and P(X=0) = 1-p.

Other kinds of discrete random variables you need to be comfortable with are uniform, binomial, and geometric random variables.

Expected value and linearity of expectation

Given two random variables X,Y on the same probability space Ω, we define a new random variable $X+Y$ on Ω by the formula

$(X+Y)(\omega) = X(\omega) + Y(\omega)$.

Going back to the previous example: Put

$$X(\mathrm{HH}) = 0,\ X(\mathrm{HT}) = 0,$$
$$X(\mathrm{TH}) = 1,\ X(\mathrm{TT}) = 1$$

(that is, 0 if the 1st toss is Heads and 1 if the 1st toss is Tails) and

$$Y(\mathrm{HH}) = 0,\ Y(\mathrm{HT}) = 1,$$
$$Y(\mathrm{TH}) = 0,\ Y(\mathrm{TT}) = 1$$

(that is, 0 if the 2nd toss is Heads and 1 if the 2nd toss is Tails).

Then $X+Y$ is the function Z satisfying

$$Z(\mathrm{HH}) = 0,\ Z(\mathrm{HT}) = 1,$$
$$Z(\mathrm{TH}) = 1,\ Z(\mathrm{TT}) = 2$$

which we recognize as the function we called Z in the previous section.

```
In[12]:= {X[HH], X[HT], X[TH], X[TT]} = {0, 0, 1, 1}

Out[12]= {0, 0, 1, 1}

In[13]:= X[HH]

Out[13]= 0

In[14]:= X[HHH]

Out[14]= X[HHH]

In[15]:= {Y[HH], Y[HT], Y[TH], Y[TT]} = {0, 1, 0, 1}

Out[15]= {0, 1, 0, 1}

In[16]:= Z[w_] := X[w] + Y[w]

In[17]:= {Z[HH], Z[HT], Z[TH], Z[TT]}

Out[17]= {0, 1, 1, 2}

In[18]:= Map[Z, {HH, HT, TH, TT}]

Out[18]= {0, 1, 1, 2}

In[19]:= Z /@ {HH, HT, TH, TT}

Out[19]= {0, 1, 1, 2}

In[20]:= AddTwoRVs := Function[{f, g}, Function[w, f[w] + g[w]]]

In[21]:= AddTwoRVs[X, Y] /@ {HH, HT, TH, TT}

Out[21]= {0, 1, 1, 2}
```

Define the <u>expected value</u> of the random variable X (in symbols, Exp(X) or E(X)) as

(1) $\quad \mathrm{E}(X) = \sum_{\omega \text{ in } \Omega} X(\omega)\, m(\omega).$

By grouping together those terms in which the value of $X(\omega)$ is the same, we get

$$\begin{aligned}(2) \qquad \mathrm{E}(X) &= \sum_a \sum_{\omega:\, X(\omega)=a} \mathrm{X}(\omega)\, m(\omega) \\ &= \sum_a \sum_{\omega:\, X(\omega)=a} a\, m(\omega) \\ &= \sum_a a \sum_{\omega:\, X(\omega)=a} m(\omega) \\ &= \sum_a a\ \mathrm{P}(X=a)\end{aligned}$$

where a ranges over all values taken

by $X(\omega)$.

Example: If X is Bernoulli with parameter p,
$E(X) = (1)(p)+(0)(1-p) = p$.

Fact: $E(X+Y)=E(X)+E(Y)$.
Proof: $E(X+Y) = \sum_\omega (X+Y)(\omega)\, m(\omega)$

$= \sum_\omega (X(\omega)+Y(\omega))\, m(\omega)$

$= \sum_\omega X(\omega)\, m(\omega) + Y(\omega)\, m(\omega)$

$= \sum_\omega X(\omega)\, m(\omega) + \sum_\omega Y(\omega)\, m(\omega)$

$= E(X) + E(Y)$.

More generally,

$$E(aX+bY) = a\,E(X) + b\,E(Y),$$

for arbitrary coefficients a,b.

This is called <u>linearity of expectation</u>.

```
In[22]:= {m[HH], m[HT], m[TH], m[TT]} = {1 / 4, 1 / 4, 1 / 4, 1 / 4}
```

Out[22]= $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$

```
In[23]:= Domain[X] = {HH, HT, TH, TT}

Out[23]= {HH, HT, TH, TT}

In[27]:= Map[X, Domain[X]]

Out[27]= {0, 0, 1, 1}

In[28]:= Map[m, Domain[X]]
```

Out[28]= $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$

```
In[24]:= Sum[X[w] m[w], {w, Domain[X]}]
```

Out[24]= $\frac{1}{2}$

```
In[29]:= ExpVal[f_] := Sum[f[w] * m[w], {w, Domain[f]}]

In[30]:= ExpVal[X]
```

Out[30]= $\frac{1}{2}$

```
In[31]:= Domain[Y] = Domain[X]

Out[31]= {HH, HT, TH, TT}

In[32]:= Domain[Z] = Domain[X];

In[35]:= Map[X, Domain[X]]

Out[35]= {0, 0, 1, 1}
```

```
In[36]:= Map[Y, Domain[Y]]
Out[36]= {0, 1, 0, 1}

In[37]:= Map[Z, Domain[Z]]
Out[37]= {0, 1, 1, 2}

In[33]:= ExpVal[Y]
```

Out[33]= $\frac{1}{2}$

```
In[34]:= ExpVal[Z]
Out[34]= 1
```

(This trick of explicitly defining the domain strikes me as kludgey; can anyone think of a better way to do this? When a function is defined by example, *Mathematica* must record the (current) domain someplace; is there a way to access this?)

```
(* For subscripts, use control-minus;
for superscripts, use control-6;
for returning to the main line, use control-space. *)
```

Multiplying random variables

Given two random variables X and Y on the probability space Ω, we define a new random variable XY on the probability space Ω by the formula $(XY)(\omega) = X(\omega)\,Y(\omega)$.

Returning to our two-coins example, XY is the function W satisfying

$$W(\mathrm{HH}) = 0,\ W(\mathrm{HT}) = 0,$$
$$W(\mathrm{TH}) = 0,\ W(\mathrm{TT}) = 1.$$

In general, it is not the case that $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$

whenever X,Y are random variables on a probability space Ω.

E.g., in our two-coins example,

$\mathrm{E}(WX) = \mathrm{E}(W) = \frac{1}{4}$ whereas

$\mathrm{E}(W)\ \mathrm{E}(X) = (\frac{1}{4})(\frac{1}{2}) = \frac{1}{8} \neq \frac{1}{4}$.

We say that the <u>random variables</u> X and Y are independent if the <u>events</u> X=a and Y=b are independent for all a and b.

That is, X and Y are independent iff

$$\mathrm{P}(X{=}a \text{ and } Y{=}b) = \mathrm{P}(X{=}a)$$

P(Y=b)
for all a,b (or equivalently "for all a in the range of X and b in the range of Y" or "for all a,b with P(X=a) and P(Y=b) positive").

Fact: When X and Y are independent,

$$E(XY) = E(X)\,E(Y).$$

For a proof, see Theorem 6.4 of Grinstead and Snell.

In our running example (tossing two

coins), X and Y are independent but X and W are not.

Variance

Definition:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \geq 0.$$

Example: With Z governed by the distribution Binomial$(2, \frac{1}{2})$ as above, $\mathrm{E}(Z)=(0)(\frac{1}{4})+(1)(\frac{1}{2})+(2)(\frac{1}{4}) = 1$ and $\mathrm{E}(Z^2)=(0)(\frac{1}{4})+(1)(\frac{1}{2})+(4)(\frac{1}{4}) = \frac{3}{2}$, so $\mathrm{Var}(Z) = \frac{3}{2} - 1 = \frac{1}{2}$.

Important special case: If X is a Bernoulli r.v. with P(X=1) = p and P(X=0) = 1-p, we have $X^2 = X$ (that is, $X^2(\omega)=X(\omega)$ for all ω), so

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= E(X) - [E(X)]^2 \\ &= p - p^2 \text{ (also written as } p(1-p)).\end{aligned}$$

Fact: If X and Y are independent r.v.'s,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Proof: See Theorem 6.8.

Example: In our two-coins example, we have

$\text{Var}(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$,

$\text{Var}(Y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, and

$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(Z)\\ &= \frac{1}{2} = \text{Var}(X) + \text{Var}(Y).\end{aligned}$$

Note by the way that

$$\text{Var}(aX) = a^2 \,\text{Var}(X),$$

so for independent random variables

X and Y,

$\text{Var}(aX+bY) = a^2\ \text{Var}(X) + b^2\ \text{Var}(Y)$.

More than two random variables

We say three random variables X_1, X_2, X_3 are independent if

$$\text{P}(X_1 = a\ \text{ and } X_2 = b\ \text{ and } X_3 = c)$$
$$= \text{P}(X_1 = a)\ \text{P}(X_2 = b)\ \text{P}(X_3 = c)$$

for all a,b,c.

Fact: If X_1, X_2, X_3 are independent, $\text{E}(X_1 X_2 X_3) = \text{E}(X_1)\ \text{E}(X_2)\ \text{E}(X_3)$.

Fact: If X_1, X_2, X_3 are independent, $\text{Var}(X_1+X_2+X_3) = \text{Var}(X_1) + \text{Var}(X_2) +$

$\text{Var}(X_3)$.

Note that the formula

$\text{E}(X_1+X_2+X_3) = \text{E}(X_1) + \text{E}(X_2) + \text{E}(X_3)$

holds in every case, whether or not the r.v.'s are independent.

That is, expected values always add, but variances typically only add when the random variables are independent.

(Non-)Example: $\Omega = \{000,011,101,110\}$, with $m(\omega) = \frac{1}{4}$ for

all ω, and with
$X(\omega)$ = the 1st bit of ω,
$Y(\omega)$ = the 2nd bit of ω,
$Z(\omega)$ = the 3rd bit of ω.
Then X and Y are independent; X and Z are independent; and Y and Z are independent (that is, the ensemble exhibits "pairwise-independence"); but X, Y, and Z (taken as a threesome) are not independent.

What is Var(X+Y+Z) in this case?

(Scroll back up and have the students take 2 minutes to compute this by hand.)

..?..

$\text{Exp}(X+Y+Z) = (0)(1/4)+(2)(3/4) = 3/2$

$\text{Exp}((X+Y+Z)^2) = (0)(1/4)+(4)(3/4) = 3$

$\text{Var}(X+Y+Z) = 3 - 9/4 = 3/4$

It's 3/4, which is also equal to the sum of the variances! Check:

$\text{Var}(X) + \text{Var}(Y) + \text{Var}(Z) = 3/4.$

What's going on here?

..?..

Fact: As long as X,Y,Z are *pairwise* independent, Var(X+Y+Z) = Var(X) + Var(Y) + Var(Z).

However, if X,Y, and Z are merely pairwise independent, we cannot conclude that E(XYZ) = E(X)E(Y)E(Z).

All of this applies straightforwardly to larger families / ensembles / sequences of random variables.

Simulation

Suppose we wish to simulate a Bernoulli random variable with parameter p, where p is a rational number, say $p = k/n$.
One way to do this is to generate a uniform random variable on the set $\{1,2,3,...,n\}$; call this random variable U. (Let $\Omega = \{1,2,...,n\}$ and let $m(\omega) = 1/n$ for all ω in Ω.) Then define a "parasitic" random variable

$$X = f(U),$$

where $f(i) = 1$ if $i \leq k$ and $f(i) = 0$ otherwise. Then X is a 0,1-valued (i.e. Bernoulli) random variable with

$$\begin{aligned}\text{Prob}(X = 1) &= \text{Prob}(f(U) = 1)\\ &= \text{Prob}(U \leq k)\\ &= 1/n + \ldots + 1/n \ (k \text{ times})\\ &= k/n\\ &= p.\end{aligned}$$

What if p isn't rational?

In *Mathematica*, this isn't a problem, since Bernoulli random variables are a built-in part of the language.

Go to the Help page and enter "Bernoulli".

```
In[38]:= Mean[RandomInteger[BernoulliDistribution[1 / 2], 1000]]
```

Out[38]= $\frac{53}{100}$

```
In[39]:= N[Mean[RandomInteger[BernoulliDistribution[Pi / 10], 1000]]]
```

Out[39]= 0.31

But what is *Mathematica* really doing? (And what can a programmer do in a language that doesn't have Bernoulli random variables built in?)

..?..

You generate a random real number between 0 and 1 (call it *U* but keep in mind that unlike our previous U it's a continuous random variable) and then define a parasitic random variable

$$X = f(U),$$

where $f(t) = 1$ if $t \le p$ and $f(t) = 0$ otherwise. Then X is a 0,1-valued (i.e. Bernoulli) random variable with

$$\begin{aligned} \text{Prob}(X = 1) &= \text{Prob}(f(U) = 1) \\ &= \text{Prob}(0 \le U \le p) \\ &= p. \end{aligned}$$

In[40]:=
```
f[x_] := If[x ≤ Pi / 10, 1, 0]
```

In[41]:=
```
N[Mean[Map[f, Table[RandomReal[], {n, 10 000}]]]]
```

Out[41]=
```
0.317
```

(Caveat: Since computer precision is finite, RandomReal[] is really generating a random rational number of the form $k/2^m$, for some fixed m, where k is a uniform random integer in $\{1,2,...,2^m\}$. So the above code is really generating a Bernoulli(p) random variable for some <u>rational</u> number p that is very close, but not equal, to $\pi/10$. More generally: on a finite precision machine, the two cases we've looked at --- rational p

vs. irrational p --- aren't all that different.)

We've just seen how to construct a Bernoulli(p) random variable from a Uniform(0,1) random variable.

Can we construct a Geometric(p) random variable from a Uniform(0,1) random variable?

```
In[42]:= f[x_] := Ceiling[Log[1 / 2, x]]

In[43]:= Map[f, Table[RandomReal[], {n, 100}]]

Out[43]= {1, 4, 2, 1, 1, 4, 2, 1, 2, 2, 1, 1, 5, 1, 1, 1, 5, 2, 1, 1, 1, 2, 4, 2, 2, 1, 1, 2, 1, 1, 1, 1, 3, 1,
  4, 3, 1, 1, 1, 1, 2, 4, 1, 1, 1, 1, 1, 1, 4, 3, 3, 2, 2, 1, 5, 4, 1, 1, 3, 1, 1, 1, 1, 2, 6, 1, 2,
  1, 1, 1, 2, 2, 1, 2, 1, 1, 2, 7, 2, 1, 2, 1, 1, 1, 4, 2, 2, 1, 2, 7, 1, 1, 1, 1, 1, 2, 2, 1, 2, 2}

In[44]:= Tally[%]

Out[44]= {{1, 54}, {4, 8}, {2, 27}, {5, 3}, {3, 5}, {6, 1}, {7, 2}}

In[45]:= Sort[%]

Out[45]= {{1, 54}, {2, 27}, {3, 5}, {4, 8}, {5, 3}, {6, 1}, {7, 2}}

In[46]:= Sort[Tally[Map[f, Table[RandomReal[], {n, 1000}]]]]

Out[46]= {{1, 494}, {2, 244}, {3, 135}, {4, 63}, {5, 24}, {6, 24}, {7, 7}, {8, 6}, {11, 2}, {16, 1}}
```

On the next homework assignment I'll ask you to generalize this to other values of *p*, and then run a simulation to check that your program is giving sensible output.

Can we construct a Normal(0,1) (aka Gaussian) random variable from a Uniform(0,1) random variable? (Note that here we're fully leaving the realm of discrete random variables;

but then we already stuck our toes over the border when we brought Uniform(0,1) random variables into the discrete realm.)

Recall the definition of the cumulative distribution function erf(x) associated with a Gaussian random variable Z:

$$\mathrm{erf}(x) = \mathrm{P}(Z \leq x).$$

(erf(x) cannot be expressed in closed form, but can be written as an integral.) erf has domain $(-\infty,\infty)$

and range (0,1). Let f be the inverse function of erf (sometimes called the probit function) with domain (0,1) and range (-∞,∞).

Claim: If U is U(0,1) (that is, if the random variable U has probability law U(0,1), i.e., is uniformly distributed on the interval [0,1]), then $f(U)$ is N(0,1) (that is, the random variable $f(U)$ has probability law N(0,1), i.e., is normally distributed with mean 0 and variance 1).

Proof: For every real number a, Prob($f(U) \leq a$) = Prob($U \leq$ erf(a)) = erf(a) = Prob($Z \leq a$), so $f(U)$ has the same cdf (cumulative distribution function) as an N(0,1) random variable.

Remark: Since there is no simple closed form for the probit function, this method is not as good as it looks. Other methods are typically used in practice (e.g. the Box-Muller method), using only arithmetic operations, trig functions, exponentials, and logs. But the probit method is conceptually the simplest method, and the method most susceptible to generalization.

The first homework problem (hints)

Suppose X and Y are Bernoulli random variables, so that their joint distribution is determined by four numbers:

$$P[X=0 \text{ and } Y=0],$$

P[X=0 and Y=1],
P[X=1 and Y=0], and
P[X=1 and Y=1].

Do we have four degrees of freedom?

No. These four probabilities must add up to 1!

That leaves just three degrees of freedom.

If we are given P[X=1] and P[Y=1] (the "marginal probabilities"), then that removes two of those degrees of freedom.

Say P[X=1] = p and P[Y=1] = q (so that P[X=0] = 1-p and P[Y=0] = 1-q).

Then, putting

P[X=1 and Y=1] = s,

we have

P[X=1 and Y=0] = p-s,

P[X=0 and Y=0] = (1-q)-(p-s),

and

P[X=0 and Y=1] = q-s.

Take p = q = 1/2. Then

P[X=0 and Y=0] = s,

$P[X=0 \text{ and } Y=1] = 1/2 - s$,

$P[X=1 \text{ and } Y=0] = 1/2 - s$, and

$P[X=1 \text{ and } Y=1] = s$,

where the only sensible values of s are $0 \le s \le 1/2$.

(The case where X and Y are independent corresponds to what value of s?

..?..

$s = 1/4$.)

Now we can compute Var[X+Y] as a function of s.

$P[X+Y = 0] = s$,

$P[X+Y = 1] = (1/2 - s) + (1/2 - s)$

$= 1 - 2s,$

$P[X+Y = 2] = s.$

$\text{Exp}[X+Y]$ = ... [have the students do it]

..?..

$(0)(s) + (1)(1-2s) + (2)(s) = 1$

(Why is this independent of s?

..?..

Linearity of expectation!

Since X and Y are Bernoulli with parameter p and q respectively, $E(X)$

= p, E(Y) = q and E(X+Y) = p+q which in our case equals 1 and in every case is a constant that doesn't depend on s.)

Exp[$(X+Y)^2$] = ... [have the students do it]

..?..

(0)(s) + (1)(1-2s) + (4)(s) = 1+2s

Var[X+Y] = Exp[$(X+Y)^2$] - $(\text{Exp}[X+Y])^2$

= (1+2s) - $(1)^2$

= 2s.

Check:

If $s=0$, Var[$X+Y$] = 0 because $X+Y$ is constantly 1;

if $s=1/4$, Var[$X+Y$] = 1/2 = Var[X] + Var[Y] because X and Y are independent; and

if $s=1/2$, Var[$X+Y$] = 1 = 4 Var[X] = Var[$2X$] = Var[$X+Y$] because $X = Y$.

The homework is similar, with $p = q = 0.4$.

Quasirandomness with rotor-routing (demo)

I demonstrated the program http://www.cs.uml.edu/~jpropp/rotor-router-model/ in its "Walk on finite graph A" mode (random walk on {0,1,2,3}).