Brownian motion

Constructing Brownian paths

Let Ω be the set of continuous real-valued functions on [0,1]. We want to pick a function f from Ω at random in accordance with Wiener measure (the probability measure underlying the mathematical definition of Brownian motion).

Suppose we have already specified the values of $f(.)$ at points $\ldots < s < u < \ldots$, and we want to specify the value of $f(.)$ at some point t in (s,u) (it could be the midpoint $(s+u)/2$ or it could be something else). Say we have chosen $f(s)=A$ and $f(u)=B$. Then the Levy construction says we should pick $f(t)$ to be a Gaussian with mean

$$\frac{u-t}{u-s}A + \frac{t-s}{u-s}B$$

and variance

$$\frac{(u-t)(t-s)}{u-s}$$

If $u = s + 1/n$ and t is the midpoint of $[s,u]$, then we should pick $f(t)$ to be a Gaussian with mean $(f(s)+f(u))/2$ and variance $1/(4n)$.

Try it on the interval [0,1]:

In[2]:=
```
Bisect[L_] :=
 (* L is the list whose kth element (1 ≤ k ≤ n) is the value of X at time k/n,
 where X is a Brownian motion.  Note that X(0) is omitted from the list. *)
 Module[{n, k, M},
  n = Length[L];
  M = Table[0, {2 n}]; (* the answer *)
  For[k = 1, k ≤ n, k++, M[[2 k]] = L[[k]]];
  M[[1]] = RandomReal[NormalDistribution[L[[1]] / 2, Sqrt[1 / (4 n)]]];
  For[k = 2, k ≤ n, k++,
   M[[2 k - 1]] = RandomReal[NormalDistribution[
       (L[[k - 1]] + L[[k]]) / 2, Sqrt[1 / (4 n)]]]];
  Return[M]]
```

In[3]:=
```
B = {RandomReal[NormalDistribution[0, 1]]}
```

Out[3]= {0.643338}

In[4]:=
```
B = Bisect[B]
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Out[4]= {0.758701, 0.643338}

In[14]:=
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B = Bisect[B]; ListPlot[B, Joined → True, PlotRange → {{0, Length[B]}, {Min[B], Max[B]}}]
```
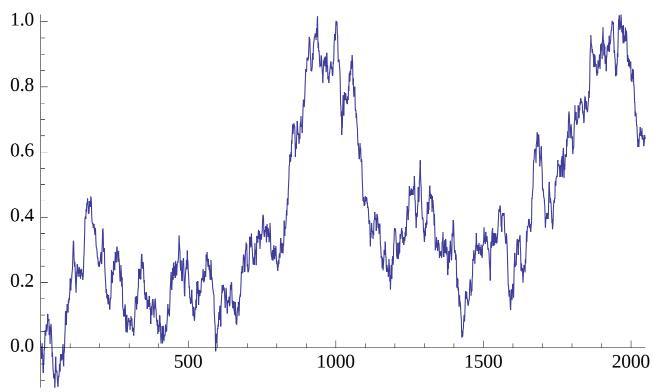

Out[14]=



For a picture of Brownian motion, see e.g. http://www.nbi.dk/~tweezer/pics/brownian-motion.jpg .

One interesting feature of this construction is that the quadratic variation

$$\sum_{k=0}^{n-1} (B_{k+1} - B_k)^2$$

converges to 1 almost surely (i.e. with probability 1):

```
In[22]:= B = Bisect[B]; Sum[(B[[k + 1]] - B[[k]])^2, {k, 1, Length[B] - 1}]
Out[22]= 1.00097
```

When we pass to the limit and make B a continuous function of time, this turns into the strange integral equation

$$\int_0^1 (B')^2 \, dt^2 = 1$$

The left hand side has two anomalous features: it features the squared time-differential dt^2 instead of the usual dt, and it features B', the time-derivative of B, which I said earlier does not exist!

These two anomalies cancel each other out if we replace the integral by Riemann sums: the jumps $B(t+\Delta t)$ - $B(t)$ are larger than we'd get if B were a differentiable function, but it's multiplied by $(\Delta t)^2$ squared which is smaller than the usual Δt. (If you like this, you'll love the Ito calculus!)

One interesting feature of Brownian motion — a kind of "Law of Large Numbers in the small" — is that on <u>any</u> time-interval $[t_1,t_2]$ the quadratic variation

$$\int_{t_1}^{t_2} (B')^2 \, dt^2$$

is almost surely equal to t_2-t_1.

So the graph of Brownian motion isn't just any old wiggly function; it's a wiggly function that bears a special kind of imprint (the quadratic variation on any subinterval is equal to the length of the subinterval), and every excerpt of this graph, no matter how small, bears this imprint on each and every part of it.

The Levy construction is not the only way to simulate Brownian motion. Something I'm trying to do in my own research is to find a way to construct Brownian motion via birth-death processes (aka random walk on $\mathbb{Z}$) by successively approximating Brownian motion on $\mathbb{R}$ by continuous-time random walks on $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/4$, ...

Two-dimensional Brownian motion

So far, the Brownian motion we've discussed is taking place in 1 dimension.

If $x(t)$ and $y(t)$ are independently doing 1-dimensional Brownian motion, then $(x(t),y(t))$ is said to be doing 2-dimensional Brownian motion.

For a picture, see e.g.

http://www.crm.umontreal.ca/~physmath/images/gallery.dir/Brownian_motion.gif .

2-dimensional Brownian motion can also be derived as a continuum limit of 2-dimensional random walk on a grid, as the grid-spacing gets finer and finer.

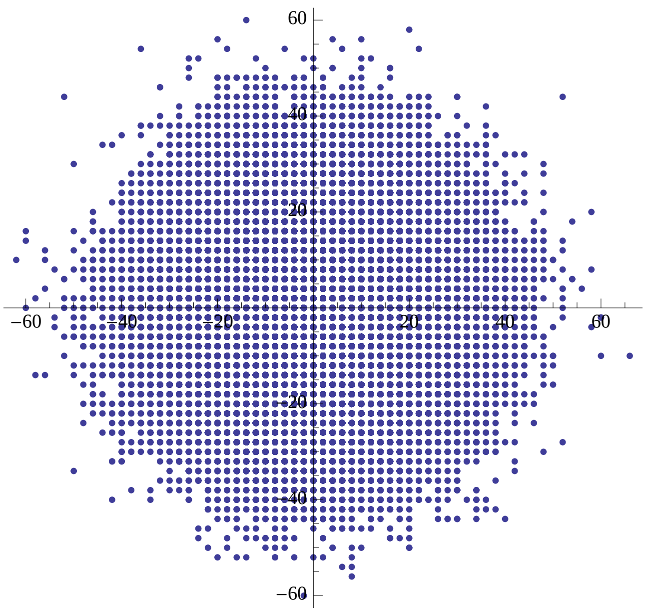
It can be shown that 2-dimensional Brownian motion is rotationally invariant. Indeed, each time-increment $(x(t+\Delta t)-x(t),\ y(t+\Delta t)-y(t))$ is just a 2-dimensional Gaussian, and one of the basic properties of a Gaussian is that a 2-dimensional Gaussian (X,Y) consisting of a pair of independent mean 0, variance 1 Gaussian random variables has rotational symmetry in $\mathbb{R}^2$.

Note that something genuinely surprising is going on here, because random walk on a 2-dimensional grid is NOT rotationally invariant (because the grid isn't!). Somehow as you make the grid finer and finer, and rescale time appropriately so that you get a sensible (albeit fractal) sort of walk-like process in the limit, the effect of the grid disappears!

One way to see this directly is to look at where you are likely to be after n steps of random walk in the grid, with n reasonably large:

In[23]:=
```
ListPlot[Table[2 {(RandomInteger[BinomialDistribution[200, 1 / 2]] - 100),
    (RandomInteger[BinomialDistribution[200, 1 / 2]] - 100)},
  {100 000}], AspectRatio → Automatic]
```

Out[23]=



Brownian motion can also be defined in higher dimensions, but one thing that's special about 2-dimensional Brownian motion is that it exhibits <u>conformal invariance</u>. That is, if D and D' are simply-connected subsets of the plane and $\varphi : D \to D'$ is a conformal (one-to-one and onto) map between those domains (i.e., an orientation-preserving map that preserves angles), then φ carries Brownian motion on D to Brownian motion on D', modulo time-parametrization. That is, the behavior of Brownian motion under a conformal map is a time-changed Brownian motion that sometimes goes "faster" and sometimes goes "slower" according to where it is, but whose itinerary, viewed as a time-ordered set of points, is statistically indistinguishable from the itinerary of Brownian motion. (I put "faster" and "slower" in quotes because Brownian motion doesn't have speed of the ordinary kind; with probability 1 it's not differentiable anywhere.)

In two dimensions, ANY two simply-connected compact subsets of the plane are related by a conformal map, so this gives a very large set of symmetries for Brownian motion -- much bigger than the symmetry group of any deterministic path (or any other sort of geometric object) could be! In higher dimensions, there are far fewer conformal maps. So there's something rather special about two-dimensional Brownian motion.