Electrical networks and random walk

Voltage and probability

(adapted from *Random Walks and Electric Networks* by Peter G. Doyle and J. Laurie Snell)

In a finite network of nodes connected by resistors, with two of the nodes joined to opposite poles of a 1-volt battery, each node x eventually settles down to a voltage v_x. If nodes x and y are connected by a resistor with resistance $R_{x,y}$, then Ohm's Law says that the amount of current $i_{x,y}$ flowing through this resistor in the direction from node x to node y is $(v_x - v_y)/R_{x,y}$. If we define the conductance as the reciprocal of resistance, then we can write this as $i_{x,y} = (v_x - v_y)C_{x,y}$.

Note that $C_{y,x} = C_{x,y}$, so that $i_{y,x} = -i_{x,y}$.

We now associate with this circuit a Markov chain whose states correspond to the nodes of the circuit.

Define the probability of a step from x to y as $p_{x,y} = C_{x,y} / C_x$, where $C_x = \sum_y C_{x,y}$. Then we can check that C_x is a stationary mass-distribution for mass-flow governed by the matrix **P**. Indeed, C_x is a reversible measure for the transition matrix **P**; that is, for all x,y we have the detailed balance condition

$$C_x \, p_{x,y} = C_y \, p_{y,x}$$

(check: this is just the assertion $C_{x,y}=C_{y,x}$). So if we normalize by defining $w(x) = C_x \, / \, C$ for all x, with $C = \sum_x C_x$, we get a stationary probability distribution **w**.

Since our circuit is connected, the Markov chain is ergodic.

Suppose the 1-volt battery is joined to nodes a and b, so that $v_a = 1$ and $v_b = 0$.

Claim (the probabilistic interpretation of voltage): For each x, v_x equals the probability that, starting from state x, the Markov chain reaches state a before state b.

Proof: By Kirchhoff's Current Law, the total net current flowing into any node x other than a or b must equal 0:

$$\sum_y C_{x,y}(v_x - v_y) = 0$$

so that

$$\begin{aligned} v_x\, C_x &= v_x \sum_y C_{x,y} = \sum_y C_{x,y} v_x \\ &= \sum_y C_{x,y} v_y. \end{aligned}$$

Dividing by C_x, we get

$$v_x = \sum_y (C_{x,y}/C_x) v_y = \sum_y p_{x,y} v_y$$

so that the voltage v_x is a harmonic function of x for all x other than a,b.

Let h_x be the probability, starting at x, that state a is reached before b. We know that h_x is harmonic away from a and b; that is,

$$h_x = \sum_y p_{x,y} h_y$$

Furthermore

$$v_a = h_a = 1$$

and

$$v_b = h_b = 0$$

Thus if we modify **P** by making a and b absorbing states, we obtain an absorbing Markov chain, and v and h are both harmonic functions for the absorbing Markov chain with the same boundary values. Hence $v = h$.

(For a probabilistic interpretation of the current $i_{x,y}$, see section 1.3.3 of Doyle & Snell.)

This relationship between voltage and probability sometimes gives an alternate way of computing probabilities, as we will shortly see.

Effective resistance

(also adapted from *Random Walks and Electric Networks*)

When we impose a voltage-difference v between nodes a and b, a voltage $v_a = v$ is established at a and a voltage $v_b = 0$ is established at b, and a current $i_a = \sum_x i_{a,x}$ will flow into the circuit from the outside source (or battery). The amount of current that flows depends upon the overall resistance in the circuit. If the voltage between a and b is multiplied by a constant, then every voltage and current in the circuit is multiplied by the same constant, so the ratio v_a / i_a is unaffected. We define the effective resistance R_{eff} between a and b as v_a / i_a and the effective conductance C_{eff} as the reciprocal quantity i_a / v_a.

Claim (the probabilistic interpretation of effective conductance): C_{eff} / C_a is the probability ("escape-probability") that the Markov chain, started at a, reaches b before it returns to a.

Proof: For simplicity (and without loss of generality) set $v_a = 1$, so that

$$C_{\text{eff}} = i_a = \sum_x i_{a,x} = \sum_x (v_a - v_x) C_{a,x}$$

$$= \sum_x v_a C_{a,x} - \sum_x v_x C_{a,x}.$$

Dividing by C_a, we get

$$C_{\text{eff}} / C_a = \sum_x v_a p_{a,x} - \sum_x v_x p_{a,x}$$

$$= 1 - \sum_x v_x p_{a,x}$$

Since each term $v_x\ p_{a,x}$ is the probability that the Markov chain takes a step from a to x and thereafter hits a before hitting b, the sum $\sum_x v_x\ p_{a,x}$ is the probability that the Markov chain takes one step and thereafter hits a before hitting b, so C_{eff} / C_a is the complementary probability, namely, the probability that the Markov chain hits b before returning to a.

One reason this interpretation can help us is that we can use formulas governing effective resistance/conductance to compute probabilities about Markov chains and random walk.

The two most useful such formulas are the series and parallel formulas:

when we add two circuits in series, their effective resistances add;

when we add two circuits in parallel, their effective conductances add.

(If I teach this topic in the future, I may ask students to **derive** these circuit laws from facts about random walk on circuits!)

Ladder graphs

These ideas can also be applied to some infinite graphs.

Consider the infinite ladder graph shown in http://faculty.uml.edu/jpropp/584/conduct.html

The rules for series and parallel combination of circuits let us show that the effective conductance between nodes A and B is $\sqrt{3}$, and $C_{\mathrm{eff}} / C_a = \sqrt{3}/3$.

To see this, first consider a one-sidedly infinite ladder graph, *L*. The infinite network is electrically equivalent to a single resistor of unknown resistance *R*. But we can also view *L* as consisting of a "smaller" (also infi - nite) copy of *L* joined to *a* and *b* by 1-ohm resistors (where a and b are also joined to one another by a 1-ohm resistor). If we replace the copy of *L* by an electri cally equivalent *R*-ohm resistor, we see that the number *R* must have the property that an *R*-ohm resistor joining *a* and *b* is electrically equivalent to a 4-node cir - cuit (with nodes *a*, *b*, a', and b') consisting of a 1-ohm resistor joining *a* and *b*, a 1-ohm resistor joining *a* and a', a 1-ohm resistor joining *b* and b', and an *R*-ohm resis - tor joining a' and b'. By the rule for series composition, the path joining *a* and *b* in this circuit by way of a' and b' has effective resistance $1+R+1 = 2+R$, i.e. effective conductance $1/(2+R)$, and since there is also a 1-ohm resistor joining *a* and *b* directly, the rule for parallel composition tells us that the whole circuit has effective conductance $1/(2+R) + 1$. But we also know that the circuit has effective resistance *R* and hence effective conductance $1/R$. Therefore $1/(2+R) + 1 = 1/R$.

We can easily solve this for *R*:

In[202]:=
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Solve[{1 / (2 + R) + 1 == 1 / R}, {R}]
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Out[202]= $\left\{\left\{R \to -1 - \sqrt{3}\right\}, \left\{R \to \sqrt{3} - 1\right\}\right\}$

The negative root is spurious, so we must have $R = \sqrt{3} - 1$. Hence the one-sided ladder circuit has effec - tive conductance

$$C = 1/(\sqrt{3} - 1) = (\sqrt{3} + 1)/2.$$

Now the doubly-infinite ladder network can be viewed as the parallel composition of a left half, a 1-ohm resis - tor in the middle, and a right half, where the left half and right half can each be viewed as the series composition of a 1-ohm resistor, an *R*-ohm resistor, and a 1-ohm resistor; so its effective conductance is $1/(2+R) + 1 + 1/(2+R) = \sqrt{3}$ and its effective resistance is $1/\sqrt{3} = \sqrt{3}/3$.

In[206]:=
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Simplify[2 / (2 + (Sqrt[3] - 1)) + 1]
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Out[206]= $\sqrt{3}$

Now turn the graph into a Markov chain by putting a unit resistor between any two nodes that are joined by an edge. Then the Markov chain is just unbiased random walk on the nodes. (Since this is a Markov chain with infinite state space, the principles we will be invoking require a more rigorous treatment than we have given so far; we were implicitly assuming that our circuits had only finitely many nodes. However, the calculation that follows can be justified.) So the probability that a random walker who starts at *A* will hit *B* before returning to *A* is $\sqrt{3}/3$.

For a proof of this that avoids electrical analysis, see http://faculty.uml.edu/jpropp/584/ladders.html .

Other geometries are possible (ladders built of triangles instead of squares, etc.); they all give nice quadratic irrationals. Rotor-walk on these graphs should be susceptible to analysis, just as in the case of the "Goldbugs" walk (although the analysis for ladder-graphs is likely to be more complicated).

Course evaluation

Any topics you'd like to have seen less of? More of? Any of at all? (E.g., branching processes.)

Should the computational part of the class be improved, and if so, how?